# Quadratic Functions

[Judy Ahrens, Pellissippi State Technical Community College]

A **quadratic function** may always be written in the form f(x)= $ax^2 + bx + c$, where a $\neq 0$. The **degree** of the function is 2 (the highest degree of its terms). Its **domain** is $(-\infty, \infty)$ because it is a polynomial function. Its graph is a (vertical) **parabola** with **y-intercept** of (0,c). Its **x-intercepts** (if any can be found by replacing f(x) with zero and solving for x. Every parabola has exactly one y-intercept, buy it may have 0,1, or 2 x-intercepts. Quadratic functions are not **one-to-one** functions since their graphs fail the **horizontal line test**. Therefore, they do not have **inverses** unless their domain is restricted in such a way that the given function is one-to-one.

**EXAMPLE**: $f(x) = 3x^2 - 2x - 1$

> The domain is $(-\infty, \infty)$, and the y-intercept is (0,1). The x-intercepts will be found below.

The sign of the coefficient "a" determine the direction of the parabola's opening. If a >0, the parabola open **upward**; if the a < 0, the parabola opens **downward**. The size of the coefficient "a" determines the **width** of the parabola. The **basic quadratic function** is $y = 1x^2$, and all other parabolas are judg4ed in relation to it. If a > 1, the graph of the function is **narrower** than the graph of the basic graph of $y = 1x^2$; if 0 < a <1, the graph is **wider** than the basic graph. Comparable statements are true for a < -1 and –1 < a <0.

**EXAMPLE**: $f(x) = 3x^2 - 2x - 1$

> The parabola opens upward since a > 0. The graph is narrower than the basic graph since a >1.

**EXAMPLE**: $f(x) = -\frac{1}{2}x^2 - 2x - 1$

> The parabola opens downward since a < 0. The graph is wider than the basic graph since –1 < a < 0.

The **vertex** of a parabola is an extremely important point. Its x-coordinate tells **where** the maximum and minimum of the function occurs, and its y-coordinate tells **what** the maximum or minimum is. The **x-coordinate** of the vertex can be found by letting $x = -\frac{b}{2a}$. We can substitute this value into the function in place of x to find the **y-**

**coordinate** of the vertex. Thus, the coordinates of the vertex are $\left[\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right]$. The **axis of symmetry**, usually just called the axis, is the vertical line passing through the vertex. Therefore, its equation is $x = -\frac{b}{2a}$.

**EXAMPLE**: $f(x) = 3x^2 - 2x - 1$

The x-coordinate of the vertex is $x = -\frac{-2}{2 \cdot 3} = \frac{1}{3}$. The y-coordinate of the vertex is $y = f\left(-\frac{b}{2a}\right) = 3\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) - 1 = -\frac{4}{3}$. Therefore, the vertex is $\left(\frac{1}{3}, -\frac{4}{3}\right)$. The axis is $x = \frac{1}{3}$. Since the parabola opens upward, it has a minimum, which is found where $x = \frac{1}{3}$. The minimum is the y-value $-\frac{4}{3}$.

**EXAMPLE**: $f(x) = -\frac{1}{2}x^2 - 2x - 1$

Following the same steps as above shows that the vertex is (-2,1), the axis is x = -2, the maximum is found where x = -2, and maximum is 1.

Once the vertex is found, the **range** can be determined. If the parabola opens upward, the is $\left[f\left(-\frac{b}{2a}, \infty\right)\right]$. The function is **increasing** on $\left(-\frac{b}{2a}, \infty\right)$ and **decreasing** on $\left(-\infty, -\frac{b}{2a}\right)$. If the parabola opens downward, the range is $\left[-\infty, f\left(-\frac{b}{2a}\right)\right]$. The function is increasing on $\left(-\infty, -\frac{b}{2a}\right)$ and decreasing on $\left(-\frac{b}{2a}, \infty\right)$.

**EXAMPLE**: $f(x) = 3x^2 - 2x - 1$

The range is $\left(-\frac{4}{3}, \infty\right)$. The function is increasing on $\left(\frac{1}{3}, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{3}\right)$.

**EXAMPLE**: $f(x) = -\frac{1}{2}x^2 - 2x - 1$

Since the vertex is $(-2,1)$, the range is $(-\infty,1)$. The function is increasing on $(-\infty,-2)$ and decreasing on $(-2,\infty)$.

The **standard** form of a quadratic function is $f(x) = a(x-h)^2 + k$. Using this form, the vertex is (h, k), the axis is x = h, and the maximum/minimum is k. The standard form is also useful for shifting the graph of $y = x^2$. If k>0, then the graph is shifted upward k units; if k<0, the graph is shifted downward k units. If h > 0, the graph is shifted to the right h units; if h < 0, the graph is shifted to the left h units. The coefficient "a" affects the graph as discussed above.

**EXAMPLE**: $f(x) = 2(x+3)^2 - 1$

The vertex if (-3,-1), the axis is x= -3, and the minimum is –1. The graph of $y = x^2$ would be shifted downward 1 unit and 3 units to the left. The opening would be narrower then the opening of the basic graph.

**EXAMPLE**: $f(x) = -0.1(x-4)^2 + 5$

The vertex is (4,5), and axis is x=4, and the maximum is 5. The graph of $y = x^2$ would be shifted upward 5 units and 4 units to the right. The opening would be wider than the opening of the basic graph.

If the vertex and any other point are given, a unique parabola is determined. The equation of the parabola can be determined by substituting the coordinates of the vertex in place of (h, k) and the coordinated of the other pint in place of (x, y) in the standard form. It is then possible to find the value of "a" and write the equation of the parabola.

**EXAMPLE**: Find the equation of the parabola with vertex (-1, 3) if it also passes through (4, -6).

Beginning with $f(x) = a(x+h)^2 + k$ substitute the coordinates of the other point to get $f(x) = a(x+1)^2 + 3$. Then substitute the coordinates of the other point to get $-6 = a(4+1)^2 + 3$. Solving for "a" yields $-\frac{9}{25}$. Therefore, the equation of the parabola is $f(x) = -\frac{9}{25}(x+1)^2 + 3$.

There are an infinite number of parabolas which will pass through any given pair of x-intercepts.

**EXAMPLE**: Find an equation of a parabola which has x-intercepts of (1, 0) and (-2, 0). This process is based on reversing the solution of a quadratic equation by factoring, which is demonstrated below. f(x) = a (x -1) (x + 2) will pass through the given points for any real value of "a" that is chosen.

**EXAMPLE**: Find an equation of a parabola which has (5, 0) as it's only x-intercept. x=5 must be considered a double root. f(x) = a(x – 5) (x – 5) will touch the x-axis at (5, 0) for any real value of "a" that is chosen.

Any three **noncollinear** points determine a unique parabola, but we do not currently have a method which will enable us to find its equation.

There are several methods of solving the quadratic equation $ax^2 + bx + c = 0$. The method of choice is **factoring**, although not all quadratic functions can be factored using integers. The next best choice is the **quadratic formula**, which works for all quadratic equations. It also has the advantage of giving exact answers. The **calculator** functions of ROOT can be used to solve quadratic equations, buy it may not give exact answers. Every quadratic equation can be solved buy **completing the square**, although this is a cumbersome method.

**EXAMPLE**: $3x^2 - 2x - 1 = 0$

$3x^2 - 2x - 1 = 0$ can be solved by factoring and applying the zero products property. (3x + 1)(x – 1) = 0 can only be true if 3x + 1 = 0 or x-1 = 0 (or both). This implies that $x = -\frac{1}{3}$ and x = 1 are the roots of the equation. The x-intercepts are $\left(-\frac{1}{3}, 0\right)$ and (1, 0).

Completing the square for the general quadratic $ax^2 + bx + c = 0$ gives the **quadratic formula**: $x = \frac{-b\sqrt{b^2 - 4ac}}{2a}$. The **discriminant** is $b^2 - 4ac$. It determines the number and types of roots of the equation. If $b^2 - 4ac > 0$, there are **two real and unequal roots**. This means that the graph has two x-intercepts. If $b^2 - 4ac = 0$, there is **one unique** real number which satisfies the equation. This is called a **double or repeated root.** In this case, the graph only touches the x-axis at a single point, which must be the vertex. Finally, if $b^2 - 4ac < 0$, there are **no real roots** of the equation, and the graph does not touch the x-axis.

**EXAMPLE**: $3x^2 - 2x - 1 = 0$

$b^2 - 4ac = 16 \Rightarrow$ there are two (different) real roots, which agrees with what we found above.

**EXAMPLE**: $x^2 + 8x + 16 = 0$

$b^2 - 4ac = 0 \Rightarrow$ there is only one distinct solution to the equation (a double root)

**EXAMPLE**: $x^2 + x + 1 = 0$

$b^2 - 4ac = -3 \Rightarrow$ there are no real solutions to the equation