Math 1020
Worksheet on Rational Functions

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A rational function is the quotient of two polynomial functions, where the denominator is not zero: $f(x)=\frac{g(x)}{h(x)}$

I. The **vertical asymptotes** are lines which the graph of the function approaches, but never touches. They can be found by setting the denominator equal to zero and solving for x. (See III for exceptions to this procedure.) If the denominator cannot equal zero, then there are no vertical asymptotes. The equations of the vertical asymptotes will always be of the form x = a. The **domain** will consist of all real numbers **except** those numbers which make the denominator equal zero.

Example: If $f(x)=\frac{3x}{x^2-9}$, then the vertical asymptotes are x = 3 and x = -3. The domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

1. Find the equations of the vertical asymptotes and the domains of each of the following functions.

a) $f(x)=\frac{x-2}{x^2-x-6}$ b) $f(x)=\frac{x^2-2x+1}{x^2-4x}$ c) $f(x)=\frac{1}{x^2+3}$

II. The **horizontal asymptote** is a line which the graph of the function approaches, but never touches **as x gets very large or very small**, i.e., as $x\to-\infty$ or $x\to\infty$. The graph **may** cross the horizontal asymptote for some real value of x. The equation of the horizontal asymptote will always be of the form y = b. Since we are examining the function as x gets large without bound, the terms of greatest importance are those with the largest exponent in both the numerator and denominator. There are three cases to be considered.

A. If the **degree of the numerator is less than the degree of the denominator,** then the horizontal asymptote is y = 0.

Example: $f(x)=\frac{3x}{x^2-9}$. Consider the equation $y=\frac{3x}{x^2}$ (found by considering only the highest degree term in numerator and denominator). It reduces to $y=\frac{3}{x}$. As x gets large without bound, the fraction gets smaller and smaller and approaches 0. Thus the equation of the asymptote is y = 0.

B. If the **degree of the numerator equals the degree of the denominator**, then the horizontal asymptote is y = the quotient of the coefficients of the highest degree terms.

Example: $f(x)=\dfrac{3x^2-3x+5}{x^2-x+6}$. Consider the equation y = $\dfrac{3x^2}{x^2}$. Since the x^2 terms will cancel, the horizontal asymptote is y = 3.

C. If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote, but there is an oblique asymptote. (See IV).

III. The range can frequently be found by using the horizontal asymptote, coupled with the graph. It will be essential to see if the graph crosses the horizontal asymptote.

A. Referring to the function in IIA, we found the horizontal asymptote to be y = 0. Can f(x) ever equal 0? If f(x)=0, then the numerator must = 0. $3x=0 \Rightarrow x=0$. Therefore, the graph of f(x) crosses its own horizontal asymptote at (0, 0). Looking at the graph of f(x) on the calculator shows that the range in this case is $(-\infty,\infty)$.

B. Referring to the function in IIB, we found the horizontal asymptote to be y = 3. Can f(x) ever equal 3? If $f(x)=\dfrac{3x^2-3x+5}{x^2-x+6}$ is to equal 3, then we see that $3x^2-3x+18=3x^2-3x+5$. This implies that 18 = 5, which is impossible. Therefore, the graph of f(x) does not cross the horizontal asymptote in this case. Looking at the graph on the calculator and using MINIMUM show that the range is [.74, 3), where the .74 is an approximation.

2. Find the equation of the horizontal asymptote, the coordinates of the point where the graph crosses the horizontal asymptote (if it exists), and the ranges of each of the following functions.

a) f(x) = $\dfrac{x^2-2x+1}{x^2-4x}$

b) f(x) = $\dfrac{1}{x^2+3}$

III. In some cases there is a "**hole**" in the graph. This happens when there is a factor which cancels out of the numerator and denominator.

Example: f(x) = $\dfrac{x-4}{x^2-6x+8}=\dfrac{x-4}{(x-4)(x-2)}=\dfrac{1}{x-2}$, if $x \neq 4$

There will be a hole (but not an asymptote) at x=4. To find the corresponding y-value, substitute 4 in place of x in the reduced form of the function. In this example, there is a hole at the point $\left(4, \frac{1}{2}\right)$.

3. Find the coordinates of the hole in the graph of f(x) = $\dfrac{x^2-3x+2}{x^2+x-6}$.

IV. If the **degree of the numerator is greater than the degree of the denominator,** then there will be an **oblique (or slant) asymptote** instead of a horizontal asymptote. To find the equation of the oblique asymptote, use long division as follows. (could use synthetic division)

Example: f(x) = $\dfrac{2x^2-x-3}{x-3}$

$$x-3\overline{\smash{)}2x^2-x-3}$$

Divide only the x into the $2x^2$; this gives 2x

$$\begin{array}{rl} & 2x \\ x-3 & \overline{\smash{)}2x^2-x-3} \end{array}$$

Multiply the (x-3) by the 2x

$$\begin{array}{rl} & 2x \\ x-3 & \overline{\smash{)}2x^2-x-3} \\ & \underline{2x^2-6x} \end{array}$$

Subtract and bring down -3

$$\begin{array}{rl} & 2x \\ x-3 & \overline{\smash{)}2x^2-x-3} \\ & \underline{2x^2-6x} \\ & \qquad\quad 5x-3 \end{array}$$

Repeat above steps until all terms are used

$$\begin{array}{rl} & 2x+5 \\ x-3 & \overline{\smash{)}2x^2-x-3} \\ & \underline{2x^2-6x} \\ & \qquad\quad 5x-5 \\ & \qquad\quad \underline{5x-15} \end{array}$$

This means that f(x) = $\left(2x+5\right)+\dfrac{12}{x-3}$. As x gets large without bound, the fraction will get smaller and smaller and approach zero. Therefore, the equation of the oblique asymptote is y = 2x+5. There are no horizontal asymptote if there is an oblique asymptote.

4. Find the equation of the oblique asymptote.

$$f\left(x\right)=\frac{2x^2-x-3}{x-2}$$

V. Find the **y-intercept**, let x=0. Some functions may not have a y-intercept. No function will ever have more than one y-intercept, since this would cause it tot fail the vertical line test for a function.

Example: f(x) = $\dfrac{x^2-4}{x^2+1}$. If x=0, then y = -4, so the y-intercept is (0,-4).

5. Find the y-intercept of each of the following functions.

a) y = $\dfrac{-3x+6}{x^2-9}$ b) f(x) = $\dfrac{x^2-2x+1}{x^2-4x}$

VI. To find the **x-intercept**, let y = 0. The only way a fraction can equal zero is if its numerator equals 0, but its denominator does not. A function may have 0, 1, or more x-intercepts.

Example: f(x) = $\dfrac{2x^2-x-3}{x-2}=0\Rightarrow 2x^2-x-3=0\Rightarrow x=-1 \text{ or } \dfrac{3}{2}$. Make sure that neither of these values causes the denominator to = 0. The x-intercepts are (-1, 0), and $\left(\dfrac{3}{2},0\right)$.

6. Find the x-intercepts of each of the following functions.

a) $f\left(x\right)=\dfrac{x^2-3x-4}{x^2+x-6}$ b) $f\left(x\right)=\dfrac{2}{\left(x+1\right)^2}$