Using Augmented Matrices to Solve Systems of Linear Equations

1. **Elementary Row Operations**

To solve the linear system $\begin{cases} x + 5y - z = -11 \\ 3z = 12 \\ 2x + 4y - 2z = 8 \end{cases}$ algebraically, these steps could be used.

All of the following operations yield a system which is ***equivalent*** to the original. (Equivalent systems have the same solution.)

Interchange equations 2 and 3 $\begin{cases} x + 5y - z = -11 \\ 2x + 4y - 2z = 8 \\ 3z = 12 \end{cases}$

Multiply equation 3 by $\frac{1}{3}$ $\begin{cases} x + 5y - z = -11 \\ 2x + 4y - 2z = 8 \\ z = 1 \end{cases}$

Multiply equation 2 by $-\frac{1}{2}$ $\begin{cases} x + 5y - z = -11 \\ -x - 2y + z = -4 \\ z = 1 \end{cases}$

Add equation 1 to 2 and replace equation 2 with the result $\begin{cases} x + 5y - z = -11 \\ 3y = -15 \\ z = 4 \end{cases}$

Multiply equation 2 by $\frac{1}{3}$ $\begin{cases} x + 5y - z = -11 \\ y = -5 \\ z = 4 \end{cases}$

Multiply equation 2 by -5 and add it to equation 1; replace equation 1 with the result $\begin{cases} x - z = 14 \\ y = -5 \\ z = 4 \end{cases}$

Add equation 3 to equation 1; replace equation 1 with the result $\begin{cases} x = 18 \\ y = -5 \\ z = 4 \end{cases}$

The solution is $(18, -5, 4)$.

2. **Operations that Produce Equivalent Systems**

a) Two equations are interchanged.
b) An equation is multiplied by a nonzero constant.
c) A constant multiple of one equation is added to another equation.

3. **Matrices**

A matrix is a rectangular array of numbers written within brackets. The **size** of a matrix is always given in terms of its ***number of rows and number of columns*** (in that order!). A ***2 x 4*** matrix has 2 rows and 4 columns. ***Square*** **matrices** have the same number of rows and columns. A matrix with a single column is called a ***column matrix,*** and a matrix with a single row is called a ***row matrix***. A square matrix with all elements on the main diagonal equal to 1 and all other elements equal to 0 is called an ***identity matrix***. The 3x3 identity matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The position of an element within a matrix is given by the ***row and column*** (in that order!) containing the element. The element a_{34} is in row 3 and column 4.

4. **Elementary Row Operations that Produce Row-Equivalent Matrices**

a) Two rows are interchanged $\quad R_i \leftrightarrow R_j$
b) A row is multiplied by a nonzero constant $\quad kR_i \to R_i$
c) A constant multiple of one row is added to another row $\quad kR_j + R_i \to R_i$

(NOTE : $\to$ means "replaces")

5. Forming an Augmented Matrix

An ***augmented*** matrix is associated with each linear system like
$$\begin{cases} x + 5y - z = -11 \\ 3z = 12 \\ 2x + 4y - 2z = 8 \end{cases}$$

The matrix to the left of the bar is called the ***coefficient*** matrix.
$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -11 \\ 0 & 0 & 3 & 12 \\ 2 & 4 & -2 & 8 \end{array}\right]$$

6. **Solving an Augmented Matrix**

To solve a system using an augmented matrix, we must use ***elementary row operations*** to change the coefficient matrix to an identity matrix.

Form the augmented matrix
$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -11 \\ 0 & 0 & 3 & 12 \\ 2 & 4 & -2 & 8 \end{array}\right]$$

Interchange rows 2 and 3
$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -11 \\ 2 & 4 & -2 & 8 \\ 0 & 0 & 3 & 12 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

Multiply row 3 by $\frac{1}{3}$	$\left[\begin{array}{ccc\|c} 1 & 5 & -1 & -11 \\ 2 & 4 & -2 & 8 \\ 0 & 0 & 1 & 4 \end{array}\right]$	$\frac{1}{3}R_3 \rightarrow R_3$
Multiply row 2 by $-\frac{1}{2}$	$\left[\begin{array}{ccc\|c} 1 & 5 & -1 & -11 \\ -1 & -2 & 1 & -4 \\ 0 & 0 & 1 & 4 \end{array}\right]$	$-\frac{1}{2}R_2 \rightarrow R_2$
Add row 1 to row 2 and replace row 2 with the result	$\left[\begin{array}{ccc\|c} 1 & 5 & -1 & -11 \\ 0 & 3 & 0 & -15 \\ 0 & 0 & 1 & 4 \end{array}\right]$	$R_1 + R_2 \rightarrow R_2$
Multiply row 2 by $\frac{1}{3}$	$\left[\begin{array}{ccc\|c} 1 & 5 & -1 & -11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{array}\right]$	$\frac{1}{3}R_2 \rightarrow R_2$
Multiply row 2 by -5 and add it to row 1; replace row 1 with the result	$\left[\begin{array}{ccc\|c} 1 & 0 & -1 & 14 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{array}\right]$	$-5R_2 + R_1 \rightarrow R_1$
Add row 3 to row 1; replace row 1 with the result	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{array}\right]$	$R_3 + R_1 \rightarrow R_1$

The solution is $(18, -5, 4)$.