The Inverse of a Matrix

Inverse Matrices

If a square matrix has an inverse, it is said to be **invertible (nonsingular)**. If A^{-1} and A are inverse matrices, then $AA^{-1} = A^{-1}A = I$ [the identity matrix]
For each of the following, use matrix multiplication to decide if matrix A and matrix B are inverses of each other. Do the multiplication by hand and show the steps.

1. Let A = $\begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$ and B = $\begin{bmatrix} -9 & \frac{7}{3} \\ -4 & 1 \end{bmatrix}$

   AB = ______________________________ and BA = ______________________________

   Are A and B inverse matrices? ______

2. Let A = $\begin{bmatrix} 5 & 6 & -2 \\ 1 & 1 & 4 \\ 2 & 2 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 6 & 26 \\ 1 & -5 & -22 \\ 0 & 0 & -1 \end{bmatrix}$

   AB = ______________________________ and BA = ______________________________

   Are A and B inverse matrices? ______

3. A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1/4 & 1/8 & -1/2 & 1/8 \\ 1/4 & 1/8 & 0 & -3/8 \\ 1/4 & -3/8 & 0 & 1/8 \\ 1/4 & 1/8 & 1/2 & 1/8 \end{bmatrix}$

   AB = ____ and

   BA = ____

   Are A and B inverse matrices? _______

Finding the Inverse Using Elementary Row Operations

Definition of Pivoting

1. Divide all entries in the pivot row by the pivot element so that the pivot element becomes 1.
2. Obtain zeros elsewhere in the pivot column by performing row operations using the pivot row.

Creating an Identity Matrix

An identity matrix I can be created by entering the proper elements as with any other matrix. The shortcut for creating the *3 x 3* identity matrix and storing it as matrix A is `MATRIX` MATH 5:identity(`ENTER` `3` `)` `ENTER` `STO+` `MATRIX` NAMES 1:[A] `ENTER`

Augmenting a Matrix with Another Matrix

Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 0 \\ -3 \end{bmatrix}$. The augmented matrix $\left[\begin{array}{ccc|c} 0 & 1 & -1 & 7 \\ 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right]$ can be created and stored created and stored as matrix C without altering the original matrices, as follows: `MATRIX` MATH 7:augment(`ENTER` `MATRIX` NAMES 1:[A] `ENTER` `,` `MATRIX` NAMES 1:[B] `ENTER` `)` `ENTER` `STO+` `MATRIX` NAMES 1:[C] `ENTER`

It is frequently useful to create an augmented matrix using an identity matrix. The augmented matrix $\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ can be formed by letting $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and using the method of the previous example.

The following is a shortcut for this special case: `MATRIX` MATH 7: augment (`ENTER` `MATRIX` NAMES 1:[A] `ENTER` `,` `MATRIX` MATH 5:identity(`ENTER` `3` `)` `ENTER` `STO+` `MATRIX` NAMES 1:[C] `ENTER`.

Finding the Inverse Using an Augmented Matrix

To find the inverse of a square matrix A (or show it does not exist), augment A with the appropriately sized identity matrix I and then using the Gauss-Jordan elimination process to try to transform the matrix to the left of the bar to the identity matrix I. If this cannot be done, then A has no inverse.

If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, then the required augmented matrix is $\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$. In order to find A^{-1}, we must use row operations until the identity matrix is to the left of the bar. The shortcut using the calculator is `MATRIX` MATH B:rref (`ENTER` `MATRIX` NAMES 1:[A] `)` `ENTER`.

The result is $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$. This means that $A^{-1} = \begin{bmatrix} -1 & 1/2 & -1/2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

To prove that our result is correct, we would need to show that $A A^{-1} = A^{-1}A = I$.

Finding the Inverse Matrix Using the Calculator

Enter $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. The calculator can find an inverse matrix directly (if it exists):

`MATRIX` NAMES 1:[A] `ENTER` `x⁻¹` `ENTER` `MATH` MATH 1:<Frac `ENTER`

4. Find the inverse of each of the following matrices, if it exists. Convert answers to fractional form.

$A = \begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$ $A^{-1} =$ ______________

$B = \begin{bmatrix} 12 & 2 \\ -5 & -10 \end{bmatrix}$ $B^{-1} =$ ______________

$C = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$ $C^{-1} =$ ______________

$D = \begin{bmatrix} 2 & -5 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ $D^{-1} =$ ______________

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E^{-1} =$ ______________

$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ $F^{-1} =$ ______________

Determinants

Although only square matrices can have inverses, there are some square matrices which do not have inverses. Each square matrix has associated with it a real number called its determinant; this number determines whether or not an inverse matrix exists.

The determinant of the *2 x 2* matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

5. Find the determinant of each of the following square matrices by hand.

$A = \begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$ $|A| =$ ________________

$B = \begin{bmatrix} 12 & 2 \\ -5 & -10 \end{bmatrix}$ $|B| =$ ________________

$C = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$ $|C| =$ ________________

Although it is possible to find determinants of larger matrices by hand, it is tedious and time-consuming. If $A = \begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$, the calculator can be used to find | A |:

`[MATRIX] MATH 1:det( [ENTER] [MATRIX] NAMES 1:[A] [)] [ENTER]`

6. Find the determinant of each of the following square matrices.

$C = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$ $|C| =$ _______

$D = \begin{bmatrix} 2 & -5 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ $|D| =$ _______

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $|E| =$ _______

$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ $|F| =$ _______

If $A = \begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$, then | A | = 3 and $A^{-1} = \begin{bmatrix} -9 & 7/3 \\ -4 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -27 & 7 \\ -12 & 3 \end{bmatrix}$. The denominator of the fraction preceding the matrix is the determinant of the original matrix. This representation has the advantage that it usually eliminates fractions from the inverse matrix. Since the determinant is in the denominator, any matrix with a zero determinant will fail to have an inverse. Such a matrix is said to be **singular**.

7. Use the determinant to classify each of the following matrices as invertible or singular.

$A = \begin{bmatrix} 12 & 5 \\ -5 & -10 \end{bmatrix}$ $|A| =$ _______, so A is ________________

$B = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$ $|B| =$ _______, so B is ________________

$C = \begin{bmatrix} 2 & -5 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ | C | = _______, so C is _______________

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | D | = _______, so D is _______________

$E = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ | E | = _______, so E is _______________

$F = \begin{bmatrix} 1 & -7 & 3 \\ 0 & 0 & 0 \\ -3 & 2 & 4 \end{bmatrix}$ | F | = _______, so F is _______________

8. Write the inverse of each of the following invertible (nonsingular) matrices in fractional form and then convert it to $\frac{1}{\det}\begin{bmatrix} \quad \end{bmatrix}$ form.

$A = \begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$ | A | = _______, A^{-1} = $= \frac{1}{\square}$

_____________ ___________

$B = \begin{bmatrix} 5 & 6 & -2 \\ 1 & 1 & 4 \\ 2 & 2 & 0 \end{bmatrix}$ | B | = _______, B^{-1} = $= \frac{1}{\square}$

______________ ____________

$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ | C | = _______, C^{-1} = $= \frac{1}{\square}$

________________ ________________

Using Matrix Inverses to Solve Equations

A linear system in standard form can be expressed as the matrix equation AX = B, where A is the matrix of coefficients of the variables, X is the column matrix of the variables, and B is the column matrix of constants appearing on the right side of the system's equations. If A is nonsingular, then the system's solution is given by X = A^{-1}B. The multiplication MUST be performed in this order!

Example: Solve the system $\begin{cases} 12x + 5y = 3 \\ -5x - 2y = 4 \end{cases}$ using matrices.

Let $A = \begin{bmatrix} 12 & 5 \\ -5 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ -5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -26 \\ 63 \end{bmatrix}$.

The solution is x = – 26 and y = 63. Check by substituting into both equations.

9. Solve each linear system following the previous example.

$$\begin{cases} 12x + 5y = -7 \\ -5x - 2y = 9 \end{cases}$$

A = B =

Write the matrix equation: $\begin{bmatrix} x \\ y \end{bmatrix} =$ (x, y) = __________

$$\begin{cases} 2x - 8y + z = 5 \\ -2x + 7y - z = -3 \\ x + y + z = 1 \end{cases}$$

A = B =

Write the matrix equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ (x, y, z) = __________

Applications of the Inverse

Use a matrix equation to solve each of the following problems.

In Collegeville, there are three pizza parlors, Tomato Pies, Say Cheese, and Crusty's, each of which offers special $50.00 catering packages to school organizations. Each parlor's package contains different amounts of pepperoni, salami, and vegi pan pizzas. The number of pounds of each type of pizza in each parlor's catering package is shown in the table below.

Parlor / Pizza	Number of pkg.	Pepperoni in lb	Salami in lb	Vegi Pan in lb	Equations
Tomato Pies	x	5	4	4	$\begin{cases} 5x + 4y + 6z = 26 \\ 4x + 5y + 6z = 25 \\ 4x + 4y + z = 14 \end{cases}$
Say Cheese	y	4	5	4	
Crusty's	z	6	6	1	
lbs required		26	25	14	

10. The student association plans to serve pizza at the Spring Parent's Day party. How many catering packages should be ordered from each pizza parlor to serve 26 lbs pepperoni, 25 lbs salami, and 14 lbs vegi pan pizzas?

A = B =

Write the matrix equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ (x, y, z) = __________

Rework the previous problem if they need 21 lbs pepperoni, 18 lbs salami, and 13 lbs vegi pan pizzas.

A = ____________________ B = ________

Write the matrix equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = ______________________________ (x, y, z) = _________

The student association is organizing a bake sale to help cover the costs of the Parent's Day party. They plan to make and sell cakes, cookies, cream pies, and brownies. The number of ounces of ingredients needed to make 32 ounces of the basic batter or filling for each dessert is shown in the table below.

Item / Ingredient	# of 32 oz batches	Flour in oz	Sugar in oz	Milk in oz	Shortening in oz	Equations
Cake batter	x	12	12	4	4	
Cookie batter	y	18	6	4	4	
Cream pie filling	z	8	6	16	2	
Brownie batter	w	8	18	0	6	
Ounces available		352	324	124	128	

How many ounces of batter or filling for each dessert can be made from 352 oz flour, 324 oz sugar, 124 oz milk, and 128 oz shortening?

A = ______________________ B = _______

Write the matrix equation: $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ = ______________________________ (x, y, z, w) = _________

Cake batter = ____ oz, cookie batter = _____ oz, cream pie filling = _____ oz, brownie batter = _____ oz.

How many ounces of batter or filling for each dessert can be made from 314 oz flour, 384 oz sugar, 188 oz milk, and 138 oz shortening?

A = ____________________ B = _______

Write the matrix equation: $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ = ______________________________ (x, y, z, w) = _________

Cake batter = ____ oz, cookie batter = _____ oz, cream pie filling = _____oz, brownie batter = _____ oz