Row Operations and Inverse Matrices on the TI-83

I. Elementary Row Operations

A. Let **A** = $\begin{bmatrix} 2 & -8 & 1 \\ -2 & 7 & -1 \end{bmatrix}$.

B. To interchange rows 1 and 2 of matrix **A**: `MATRIX` `MATH C:rowSwap(` `ENTER` `MATRIX` `NAMES 1:[A]` `,` `1` `,` `2` `)` `ENTER`.

The result is $\begin{bmatrix} -2 & 7 & -1 \\ 2 & -8 & 1 \end{bmatrix}$.

C. To add rows 1 and 2 of matrix **A**: `MATRIX` `MATH D:row+(` `ENTER` `MATRIX` `NAMES 1:[A]` `,` `1` `,` `2` `)` `ENTER`.

The result is $\begin{bmatrix} 2 & -8 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. Row 1 remains unchanged, while the result of the row operation replaces row 2.

D. To multiply row 1 of matrix **A** by 4: `MATRIX` `MATH E:*row(` `4` `,` `MATRIX` `NAMES 1:[A]` `,` `1` `)` `ENTER`.

The result is $\begin{bmatrix} 8 & -32 & 4 \\ -2 & 7 & -1 \end{bmatrix}$.

E. To multiply row 1 of matrix **A** by 5 and add it to row 2: `MATRIX` `MATH F:*row+(` `ENTER` `5` `,` `MATRIX` `NAMES 1:[A]` `,` `1` `,` `2` `)` `ENTER`

The result is $\begin{bmatrix} 2 & -8 & 1 \\ -2 & -33 & 4 \end{bmatrix}$. Row 1 remains unchanged, while the result of the row operation replaces row 2.

F. The result of row operations is displayed on the screen, but it is not stored! It is usually desirable to store the result for further operations. To store the result from the last operation performed as **F**: `STO+` `MATRIX` `NAMES 6:[F]` `ENTER`.

II. Inverse Matrices

A. If a square matrix has an inverse, it is said to be **invertible (nonsingular)**. If $\mathbf{A}^{-1}$ and **A** are inverse matrices, then $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

B. Let **A** = $\begin{bmatrix} 5 & 6 & -2 \\ 1 & 1 & 4 \\ 2 & 2 & 0 \end{bmatrix}$ and **B** = $\begin{bmatrix} -1 & 6 & 26 \\ 1 & -5 & -22 \\ 0 & 0 & -1 \end{bmatrix}$.

Since **AB** = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 8 \end{bmatrix}$ and **BA** = $\begin{bmatrix} 53 & 52 & 26 \\ -44 & -43 & -22 \\ -2 & -2 & 0 \end{bmatrix}$, **A** and **B** are not inverses.

C. Let **A** = $\begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$ and **B** = $\begin{bmatrix} -9 & 7/3 \\ -4 & 1 \end{bmatrix}$.
Since **AB** = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and **BA** = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **A** and **B** are inverse matrices.

III. Creating an Identity Matrix

A. An identity matrix **I** can be created by entering the proper elements as with any other matrix.

B. The shortcut for creating the *3x3* identity matrix and storing it as matrix **A** is [MATRIX] `MATH 5:identity(` [3] [)] [ENTER] [STO+] [MATRIX] `NAMES 1:[A]` [ENTER].

IV. Augmenting a Matrix with Another Matrix

A. Let **A** = $\begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and **B** = $\begin{bmatrix} 7 \\ 0 \\ -3 \end{bmatrix}$. The augmented matrix $\left[\begin{array}{ccc|c} 0 & 1 & -1 & 7 \\ 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right]$.
can be created and stored as matrix **C** without altering the original matrices, as follows: [MATRIX] MATH 7:augment([MATRIX] NAMES 1:[A] [,] [MATRIX] NAMES 2:[B] [ENTER] [)] [ENTER] [STO+] [MATRIX] NAMES 1:[C] [ENTER].

B. It is frequently useful to create an augmented matrix using an identity matrix. The augmented matrix $\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ can be formed by letting A = $\begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and using the method of the previous example.

C. A shortcut for augmenting a square matrix with an identity matrix: [MATRIX] `MATH 7:augment(` [MATRIX] `NAMES 1:[A]` [,] [MATRIX] `MATH 5:identity(` [3] [)] [ENTER] [STO+] [MATRIX] `NAMES 3:[C]` [ENTER].

V. Gauss-Jordan Elimination

A. Perform any row operation which will yield element $a_{1,1} = 1$. Avoid creating fractions whenever possible!

B. Use the first row and row operations to convert all other elements in column 1 to zeros.

C. Multiply row 2 by a scalar to yield element $a_{2,2} = 1$. Sometimes there are other options available, but be careful that you do not change any elements in column 1!

D. Use row 2 and row operations to covert all other elements in column 2 to zeros.

E. Repeat this process until all columns have been converted.

VI. Finding an Inverse Matrix Using an Augmented Matrix and Elementary Row Operations

A. If a square matrix **A** has an inverse, it can be found by augmenting **A** with the appropriately sized identity matrix **I** and then using the Gauss-Jordan elimination method to transform the matrix to the left of the bar into the identity matrix **I**. If this cannot be done, then **A** is singular and has no inverse.

B. If $\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, then the required augmented matrix is $\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$.

1. $\left[\begin{array}{ccc|ccc} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $\quad R_1 : R_2$

   $\left[\begin{array}{ccc|ccc} 1 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $\quad \frac{1}{2}R_1$ **6** R_1

   The rest of the elements in column 1 already equal zero.

2. Element $a_{2,2}$ already equals 1.

   $\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & -1 & 1/2 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $\quad -1R_2 + R_1$ **6** R_1

   Column 2 is now completed.

3. Element $a_{3,3}$ already equals 1.

   $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1/2 & -1/2 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $\quad -\frac{1}{2}R_3 + R_1$ **6** R_1

   $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $\quad R_3 + R_2$ **6** R_2

   Column 3 is now completed.

4. This means that $A^{-1} = \begin{bmatrix} -1 & 1/2 & -1/2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

5. To prove the result is correct, show that $\mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

VII. Shortcuts for Finding an Inverse Matrix

A. To find the row reduced form of the augmented matrix **A** from above: `MATRIX` `MATH B:rref(` `MATRIX` NAMES 1:[A] `)` `ENTER`.

B. To find $\mathbf{A^{-1}}$ directly, begin with **A** (not augmented): `MATRIX` NAMES 1:[A] `x⁻¹` `ENTER`.

C. If $\mathbf{A} = \begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$, then $\mathbf{A^{-1}} = \begin{bmatrix} -9 & 7/3 \\ -4 & 1 \end{bmatrix}$.

D. If $\mathbf{C} = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$, then $\mathbf{C^{-1}}$ does not exist.

E. If $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\mathbf{E^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

VIII. Determinants

A. Although only square matrices can have inverses, there are some square matrices which do not have inverses. Each square matrix has associated with it a real number called its determinant, and the determinant determines whether or not an inverse matrix exists.

B. The determinant of the *2x2* matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is equal to ad – bc.

1. If $\mathbf{A} = \begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$, then |A| = 3(– 27) – 12(– 7) = 3.

2. If $\mathbf{B} = \begin{bmatrix} 12 & 5 \\ -5 & -10 \end{bmatrix}$, then |B| = 12(– 10) – (– 5)(5) = – 95.

3. If **C** = $\begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$, then |C| = 3(4) – (– 2)(– 6) = 0.

C. Although it is possible to find determinants of larger matrices by hand, it can be tedious and time-consuming. If **A** = $\begin{bmatrix} 3 & -7 \\ 12 & -27 \end{bmatrix}$, the calculator can be used to find |A|: `MATRIX` `MATH 1:det(` `MATRIX` `NAMES 1:[A]` `)` `ENTER`.

D. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

E. Determinants may be used to write inverse matrices.

1. $\begin{vmatrix} 3 & -7 \\ 12 & -27 \end{vmatrix} = 3$, and $A^{-1} = \begin{bmatrix} -9 & \frac{7}{3} \\ -4 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -27 & 7 \\ -12 & 3 \end{bmatrix}$

   a. The denominator of the scalar is the determinant of the original matrix.
   b. This representation usually eliminates fractions from the inverse matrix.
   c. Since the determinant is in the denominator of the scalar, any matrix with a zero determinant will fail to have an inverse. Such a matrix is **singular**.

2. Since $\begin{vmatrix} 2 & -5 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 2$, so $\begin{bmatrix} 2 & -5 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible.

$$\begin{bmatrix} 2 & -5 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{9}{2} & -\frac{7}{2} & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 8 & 6 & -2 \\ 1 & 1 & 0 \\ 9 & -7 & 4 \end{bmatrix}.$$

IX. Using Matrix Inverses to Solve Equations

A. A linear system in standard form can be expressed as the matrix equation **AX** = **B**, where **A** is the matrix of coefficients of the variables, **X** is the column matrix of the variables, and **B** is the column matrix of constants appearing to the right of the equal signs.

B. If A is nonsingular, then the system's solution is given by $\mathbf{X = A^{-1}B}$. The inverse matrix $\mathbf{A}^{-1}$ must be placed to the left of **B** as shown!

C. To solve the system $\begin{cases} 12 + 5y = 3 \\ -5x - 2y = 4 \end{cases}$ using matrices.

1. $\mathbf{A} = \begin{bmatrix} 12 & 5 \\ -5 & -2 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\mathbf{Y}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ -5 & -2 \end{bmatrix}^{\mathbf{-1}} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -26 \\ 63 \end{bmatrix}$.
2. This means the solution is x = – 26 and y = 63. Check by substituting these values into both equations.

D. If the coefficient matrix **A** is singular, then the system has no solution.

E. For $\begin{cases} 2x - 8y + z = 5 \\ -2x + 7y - z = -3 \\ x + y + z = 1 \end{cases}$: $\mathbf{A} = \begin{bmatrix} 2 & -8 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

1. $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 2 & -8 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{\mathbf{-1}} \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 & -9 & -1 \\ -1 & -1 & 0 \\ 9 & 10 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -14 \\ -2 \\ 17 \end{bmatrix}$.

2. This means that x = – 14, y = – 2 and z = 17.

X. Cramer's Method

A. Another method for solving systems of simultaneous equations uses determinants. The system must be "square". [Number of equations = number of variables]

B. For $\begin{cases} 2x - 8y + z = 5 \\ -2x + 7y - z = -3 \\ x + y + z = 1 \end{cases}$:

1. $x = \dfrac{\begin{vmatrix} 5 & -8 & 1 \\ -3 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -8 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix}} = \dfrac{14}{-1} = -14.$

The determinant on the bottom is the coefficient determinant.
The determinant on top is the same as the coefficient determinant, except for the "x" column which is the column to the right of the equal signs.

2. $y = \dfrac{\begin{vmatrix} 2 & 5 & 1 \\ -2 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -8 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix}} = \dfrac{2}{-1} = -2.$

The determinant on the bottom is the coefficient determinant.
The determinant on top is the same as the coefficient determinant, except for the “y” column which is the column to the right of the equal signs.

$$3.\quad z = \frac{\begin{vmatrix} 2 & -8 & 5 \\ -2 & 7 & -3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -8 & 1 \\ -2 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{-17}{-1} = 17.$$

The determinant on the bottom is the coefficient determinant.
The determinant on top is the same as the coefficient determinant, except for the “z” column which is the column to the right of the equal signs.

XI. Solving Applications using an Inverse Matrix

The following application can be solved using a matrix equation or Cramer’s method.

In Collegeville, there are three pizza parlors, Tomato Pies, Say Cheese, and Crusty’s, each of which offers special $50.00 catering packages to school organizations. Each parlor’s package contains different amounts of pepperoni, salami, and vegi pan pizzas. The number of pounds of each type of pizza in each parlor’s catering package is shown in the table below.

<table>
<tr><th>Parlor / Pizza</th><th>Number of pkg.</th><th>Pepperoni in lb</th><th>Salami in lb</th><th>Vegi Pan in lb</th><th>Equations</th></tr>
<tr><td>Tomato Pies</td><td>x</td><td>5</td><td>4</td><td>4</td><td rowspan="4">$\begin{cases} 5x + 4y + 6z = 26 \\ 4x + 5y + 6z = 25 \\ 4x + 4y + z = 14 \end{cases}$</td></tr>
<tr><td>Say Cheese</td><td>y</td><td>4</td><td>5</td><td>4</td></tr>
<tr><td>Crusty’s</td><td>z</td><td>6</td><td>6</td><td>1</td></tr>
<tr><td colspan="2">lbs required</td><td>26</td><td>25</td><td>14</td></tr>
</table>

The student association plans to serve pizza at the Spring Parent’s Day party. How many catering packages should be ordered from each pizza parlor to serve 6 lbs pepperoni, 25 lbs salami, and 14 lbs vegi pan pizzas?