RATIONAL FUNCTIONS AND THEIR GRAPHS

I. f is a ***rational function*** if $f(x) = \dfrac{g(x)}{h(x)}$, where g(x) and h(x) are polynomials, $h(x) \neq 0$.

[HINT: Look for a variable in the denominator.]

II. The ***domain*** of a rational function consists of all real numbers <u>except</u> the zeros of the denominator. To find the ***zeros*** of the denominator, set it equal to 0, and solve for x.

A. $f(x) = \dfrac{x - 2}{x^2 - x - 6}$

1. $x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = 3, -2$
2. Domain of f is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

B. If the denominator does not equal 0 for any real number, then the domain is $(-\infty, \infty)$

III. The line x = a is a ***vertical asymptote*** for the graph of f(x) if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ from either side. This means that as x gets closer and closer to a, the graph of f(x) gets closer and closer to the line x = a, without touching it.

A. In the example above, the vertical asymptotes are x = 3 and x = –2.

B. Although there is not a limit to the number of vertical asymptotes a rational function may have, it is also possible that a rational function may have no vertical asymptote.

IV. The equation(s) of the vertical asymptote(s) are x = a, x = b, ..., where a, b, ... are zeros of the denominator. If the denominator cannot equal 0, then there are no vertical asymptotes. If a factor of the denominator cancels out, there will be no vertical asymptote at that zero.

A. $f(x) = \dfrac{-3x^2}{x^2 + 1}$ has no vertical asymptotes since $x^2 + 1 \neq 0$

B. $f(x) = \dfrac{2x^2 + x - 6}{x^2 + 3x + 2} = \dfrac{(2x - 3)(x + 2)}{(x + 1)(x + 2)}$

1. f has one vertical asymptote, x = –1.
2. Since (x + 2) cancels out, there is no asymptote at x = –2.
3. However, there is a **hole** in the graph at x = –2. [–2 is not in domain of f(x).]

C. The graph can ***never*** touch a vertical asymptote.

V. The line y = c is a ***horizontal asymptote*** for the graph of f(x) if $f(x) \to c$ as $x \to \infty$ or $f(x) \to c$ as $x \to -\infty$. This means that as x gets very large (positively or negatively) some portion of the graph of f(x) approaches the line y = c without touching it. However, some other portion of the graph ***may touch or even cross*** the line y = c.

VI. Since the horizontal asymptote is determined by the behavior of the function as $|x|$ becomes very large, we consider only the terms of highest degree in the numerator and denominator. We will consider three cases:

A. Degree of numerator is less than degree of denominator: $f(x) = \dfrac{3}{x - 4}$

1. As x gets larger and larger without bound, the y = f(x) gets smaller and smaller.
2. The horizontal asymptote is ***y = 0***.

B. Degree of numerator equals degree of denominator: $y = \dfrac{-3x^2 + 8}{x^2 - 4x}$

1. As x gets larger and larger without bound, the dominating terms are those with the highest degrees.
2. As x **6 4**, y **.** $\dfrac{-3x^2}{x^2} = -3$
3. The horizontal asymptote is $y = -3$

C. Degree of numerator is one larger than degree of denominator: $f(x) = \dfrac{x^3 - x^2 - 6}{x^2 - 4}$

1. As x becomes large without bound, the numerator increases faster than the the denominator.
2. There is no horizontal asymptote.
3. $$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad x - 1 \\ \hline x^2 - 4 & x^3 - x^2 + 0x - 6 \\ & \underline{x^3 \quad\quad\quad - 4x} \\ & \quad - x^2 + 4x - 6 \\ & \quad \underline{- x^2 \quad\quad\quad + 4} \\ & \quad\quad\quad\quad 4x - 10 \end{array}$$ **Y** $f(x) = x - 1 + \dfrac{4x - 10}{x^2 - 4}$
4. The ***oblique or slant asymptote*** is $y = x - 1$

VII. The range of a rational function can be difficult to determine. It is frequently helpful to consider both the equation and the graph.

A. The range of a rational function with ***linear numerators and denominators*** can be readily determined.

B. $g(x) = \dfrac{4 - 10x}{5x + 1}$ has a horizontal asymptote of $y = -2$, which implies that its range is (– **4**, – 2) ^ (– 2, **4**).

VIII. The domain of $f(x) = \dfrac{3x^2 - 3x - 6}{x^2 + 8x + 16}$ is (–**4**, –4) ^ (–4, **4**) and its horizontal asymptote is y = 3.

A. Can f(x) = 3?

B. $f(x) = \dfrac{3x^2 - 3x - 6}{x^2 + 8x + 16} = 3$ implies $3x^2 - 3x - 6 = 3(x^2 + 8x + 16)$. Solving this equation shows that x = – 2. Note that –2 is in the domain of the function.

C. Thus, y = f(x) = 3 when x = –2. The graph ***intersects the horizontal asymptote*** y = 3 at the point (–2, 3).

IX. Sketching the graph of a rational function: $f(x) = \dfrac{x^2 - x - 6}{x^2 - 4}$

A. $f(x) = \dfrac{x^2 - x - 6}{x^2 - 4} = \dfrac{(x + 2)(x - 3)}{(x + 2)(x - 2} = \dfrac{x - 3}{x - 2}$ **IF** x **…** –2. There is a hole at

$(-2, f(-2)) = \left(-2, \dfrac{5}{4}\right)$. Plot this point as an ***open*** circle.

B. The vertical asymptote is x = 2. Plot this as a dashed line.

C. The domain is (–**4**, –2) ^ (–2, 2) ^ (2, **4**).

D. The horizontal asymptote is y = 1. Plot this as a dashed line.

E. Although $\dfrac{x^2 - x - 6}{x^2 - 4} = 1$ seems to imply that x = –2, we must remember that –2 is not in the domain of the function. Therefore, the graph does not intersect the horizontal asymptote.

F. The range is (–**4**, 1) ^ (1, **4**).

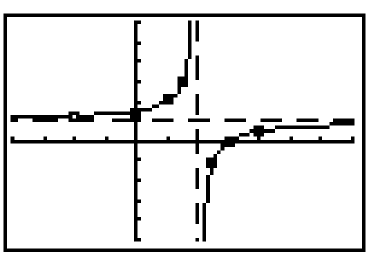
G. Letting x = 0 shows that (0, 1.5) is the y-intercept. Plot this point.

H. Letting y = 0 shows that (3, 0) is the x-intercept. Plot this point.

I. Find enough additional points to enable you to draw a reasonable graph. Make use of the end behavior as you draw the graph.

x	–1	1	1.5	2.5	4
y	4/3	2	3	–1	1/2

J.



X. Summary of steps used in sketching the graph of a rational function.

A. Factor numerator and denominator completely. If there is a common factor, (x – a), cancel it. NOTE: There will now be a hole in the graph at x = a. To find the y-coordinate of the hole, find f(a) in ***reduced*** form. Draw an open circle at (a, f(a)).
B. Find any x-intercepts by letting y = 0. Plot these points.
C. Find the zeros of the denominator. Sketch vertical asymptotes with dashed lines.
D. Find the y-intercept (if any) by letting x = 0. Plot this point.
E. Determine if there is a horizontal asymptote. If so, sketch it as a dashed line.
F. Check to see if the graph intersects the horizontal asymptote. If so, plot this point.
G. Find the equation of the oblique asymptote (if any). Sketch with a dashed line.
H. Plot additional key points. Sketch the graph using the plotted points and the asymptotes as guides.
I. Check using your calculator in dot MODE.