PROOF BY MATHEMATICAL INDUCTION

To prove: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ [n is an integer ≥ 1]

Step 1: Show that the formula is true for the first 3 integers [Ahrens' version]
[Strictly speaking, the principle of mathematical induction only requires that the formula be shown to work for n = 1.]

If $n = 1$: $1^2 = 1$ and $\sum_{k=1}^{1} k^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$

If $n = 2$: $1^2 + 2^2 = 5$ and $\sum_{k=1}^{2} k^2 = \frac{2(2+1)(2(2)+1)}{6} = \frac{30}{6} = 5$

If $n = 3$: $1^2 + 2^2 + 3^2 = 14$ and $\sum_{k=1}^{3} k^2 = \frac{3(3+1)(2(3)+1)}{6} = \frac{84}{6} = 14$

Step 2: Assume that the formula is true for some integer a > 1

If $n = a$: Assume $\sum_{k=1}^{a} k^2 = \frac{a(a+1)(2a+1)}{6}$

Step 3: Prove that the above assumption implies that the formula is also true for the next integer a+1

If $n = a+1$: $\sum_{k=1}^{a+1} k^2 = \frac{a(a+1)(2a+1)}{6} + (a+1)^2$

$$= \frac{a(a+1)(2a+1) + 6(a+1)^2}{6} = \frac{(a+1)[a(2a+1)+6(a+1)]}{6}$$

$$= \frac{(a+1)(2a^2+a+6a+6)}{6} = \frac{a+1)(2a^2+7a+6)}{6}$$

$$= \frac{(a+1)(a+2)(2a+3)}{6} = \frac{(a+1)[(a+1)+1][2(a+1)+1]}{6}$$

We observe that this is the just the original formula where n = a+1!

Step 4: Conclude that the formula is true for every integer n ≥ 1

Therefore, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$.

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