INDETERMINATE FORMS AND L'HOSPITAL'S RULE

OBJECTIVE: Use l'Hospital's Rule to find limits of indeterminate forms

l'Hospital's Rule is a handy shortcut for finding the limit of **some functions** $l(x)$ which are of the form $l(x) = \frac{f(x)}{g(x)}$. We first determine if the given function meets the prerequisites for l'Hospital's Rule.

- Necessary conditions for applying l'Hospital's Rule
  - f(x) and g(x) must both be **differentiable functions** near a (not necessarily **at a**)
  - $g'(x) \neq 0$ near a (except possibly at a)
  - $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$; $\lim\limits_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\pm\frac{\infty}{\infty}$; these are called **indeterminate forms**

- IF l'Hospital's Rule applies, then the result of applying it is:
  - $\lim\limits_{x\to a} \frac{f(x)}{g(x)} \overset{H}{=} \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ Note the use of $\overset{H}{=}$; this is not an equality in the normal sense of the word.
  - Always simplify after applying l'Hospital's Rule. If the **simplified result meets all the necessary conditions**, then l'Hospital's Rule may be applied again. Always ***simplify*** before applying the rule again!
  - Using l'Hospital's Rule when it doesn't apply is **wrong**, even though it may give the "right answer".

Example 1: Find $\lim\limits_{x\to\infty} \frac{4x^2 - 3x + 6}{5 - 2x^2}$

- We already know one way to find this limit!

$$\lim_{x\to\infty} \frac{4x^2 - 3x + 6}{5 - 2x^2} = \lim_{x\to\infty} \frac{\frac{4x^2}{x^2} - \frac{3x}{x^2} + \frac{6}{x^2}}{\frac{5}{x^2} - \frac{2x^2}{x^2}} = \lim_{x\to\infty} \frac{4 - \overset{\nearrow 0}{\frac{3}{x}} + \overset{\nearrow 0}{\frac{6}{x^2}}}{\underset{\searrow 0}{\frac{5}{x^2}} - 2} = \frac{4}{-2} = -2$$

- **Check** to see if l'Hospital's Rule applies:
  - The numerator and denominator of the function are both polynomials, which are differentiable everywhere
  - The derivative of the denominator is not equal to zero as x increases without bound
  - $\lim\limits_{x\to\infty} \frac{4x^2 - 3x + 6}{5 - 2x^2} = \frac{\infty}{\infty}$, which is one of the indeterminant forms
  - Therefore, l'Hospital's Rule applies!

- $\lim_{x\to\infty} \frac{4x^2 - 3x + 6}{5 - 2x^2} \overset{H}{=} \lim_{x\to\infty} \frac{8x - 3}{-4x}$

  - We can show that the new function $m(x) = \frac{8x - 3}{-4x}$ meets all the conditions of l'Hospital's Rule and that $\lim_{x\to\infty} \frac{8x - 3}{-4x} = \frac{\infty}{\infty}$.

  - So, $\lim_{x\to\infty} \frac{4x^2 - 3x + 6}{5 - 2x^2} \overset{H}{=} \lim_{x\to\infty} \frac{8x - 3}{-4x} \overset{H}{=} \lim_{x\to\infty} \frac{8}{-4} = -2$

The previous example is not very impressive because we already knew how to find the derivative. However, l'Hospital's Rule will enable us to find other types of limits.

Example 2: Find $\lim_{x\to 0} \frac{\sin x}{x + x^2}$

- **<u>Check</u>** to see if l'Hospital's Rule applies:

  - The numerator and denominator are both differentiable functions near 0 (and actually at 0, in this case)
  - The derivative of the denominator is 0 at x = 0, but it is **<u>not zero in the neighborhood of 0</u>**
  - $\lim_{x\to 0} \frac{\sin x}{x + x^2} = \frac{0}{0}$, which is one of the **<u>indeterminate forms</u>**
  - Therefore, l'Hospital's Rule applies

- Thus, $\lim_{x\to 0} \frac{\sin x}{x + x^2} \overset{H}{=} \lim_{x\to 0} \frac{\cos x}{1 + 2x} = \frac{1}{1} = 1$

Example 3: Find $\lim_{x\to\infty} \frac{e^{2x^2}}{x^3}$

- It can be shown that l'Hospital's Rule applies to this limit, and
- $\lim_{x\to\infty} \frac{e^{2x^2}}{x^3} = \frac{\infty}{\infty}$

- $\lim_{x\to\infty} \frac{e^{2x^2}}{x^3} \overset{H}{=} \lim_{x\to\infty} \frac{4xe^{2x^2}}{3x^2} = \lim_{x\to\infty} \frac{4e^{2x^2}}{3x}$ **<u>always reduce</u>**

- l'Hospital's Rule applies again and $\lim_{x\to\infty} \frac{4e^{2x^2}}{3x} = \frac{\infty}{\infty}$
- $\lim_{x\to\infty} \frac{e^{2x^2}}{x^3} \overset{H}{=} \lim_{x\to\infty} \frac{4xe^{2x^2}}{3x^2} = \lim_{x\to\infty} \frac{4e^{2x^2}}{3x} \overset{H}{=} \lim_{x\to\infty} \frac{16xe^{2x^2}}{3} = \infty$

Example 4: Find $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{x^2}$

- $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{x^2} \overset{H}{=} \lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{2x} \overset{H}{=} \lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{2} = \dfrac{2}{2} = 1$
  - But this is **<u>WRONG</u>** because l'Hospital's Rule did not apply to the original function!
- The correct way to find this limit: $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{x^2} = \lim\limits_{x\to 0} \left[(e^x + e^{-x}) \dfrac{1}{x^2} \right] = 2(\infty) = \infty$
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Find each of the following limits

- $\lim\limits_{x\to 0} \dfrac{x + \tan x}{\sin x}$

- $\lim\limits_{x\to 0} \dfrac{5^x - 3^x}{x}$

- $\lim\limits_{x\to 0} \dfrac{\cos mx - \cos nx}{x^2}$

Example 5: Find $\lim\limits_{x\to -\infty} xe^x$

- $\lim\limits_{x\to -\infty} xe^x = (-\infty)(0)$ This is an **<u>indeterminate product</u>** of type $(\pm\infty)(0)$
- We can rewrite this limit to obtain a fractional indeterminate form as follows:
  - Since $\lim\limits_{x\to -\infty} xe^x = \lim\limits_{x\to -\infty} \dfrac{x}{e^{-x}}$, l'Hospital's Rule applies and
  - $\lim\limits_{x\to -\infty} \dfrac{x}{e^{-x}} = \dfrac{-\infty}{\infty}$
  - Therefore, $\lim\limits_{x\to -\infty} xe^x = \lim\limits_{x\to -\infty} \dfrac{x}{e^{-x}} \overset{H}{=} \lim\limits_{x\to -\infty} \dfrac{1}{-e^{-x}} = 0$

Find $\lim_{x \to \infty} x^3 e^{-x^2}$

Example 6: Find $\lim_{x \to 0} (\csc x - \cot x)$

- $\lim_{x \to 0} (\csc x - \cot x) = \infty - \infty$ This is an **<u>indeterminate difference</u>** of type $\infty - \infty$:
- $\lim_{x \to 0} (\csc x - \cot x) = \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x \to 0} \left(\frac{1 - \cos x}{\sin x} \right)$
- Now we can see that l'Hospital's Rule applies and
- $\lim_{x \to 0} \left(\frac{1 - \cos x}{\sin x} \right) = \frac{0}{0}$
- Therefore, $\lim_{x \to 0} (\csc x - \cot x) = \lim_{x \to 0} \left(\frac{1 - \cos x}{\sin x} \right) \overset{H}{=} \lim_{x \to 0} \frac{\sin x}{\cos x} = \lim_{x \to 0} \tan x = 0$

Find $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x - 1} \right)$

<u>Indeterminate power types</u> include 0^0, ∞^0, and 1^∞. They are discussed on page 302.

Solutions

- $\lim\limits_{x\to 0} \dfrac{x+\tan x}{\sin x} = \dfrac{0}{0}$

$$\lim_{x\to 0} \frac{x+\tan x}{\sin x} \overset{H}{=} \lim_{x\to 0} \frac{1+\sec^2 x}{\cos x} = \frac{1+1}{1} = 2$$

- $\lim\limits_{x\to 0} \dfrac{5^x - 3^x}{x} = \dfrac{0}{0}$

$$\lim_{x\to 0} \frac{5^x - 3^x}{x} \overset{H}{=} \lim_{x\to 0} \frac{5^x \ln 5 - 3^x \ln 3}{1} = \ln 5 - \ln 3 = \ln\frac{5}{3}$$

- $\lim\limits_{x\to 0} \dfrac{\cos mx - \cos nx}{x^2} = \dfrac{0}{0}$

$$\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2} \overset{H}{=} \lim_{x\to 0} \frac{-m\sin mx + n\sin nx}{2x} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2} \overset{H}{=} \lim_{x\to 0} \frac{-m\sin mx + n\sin nx}{2x} \overset{H}{=} \lim_{x\to 0} \frac{-m^2\cos mx + n^2\cos nx}{2} =$$

$$\frac{-m^2 + n^2}{2} = \frac{1}{2}(n^2 - m^2)$$

- $\lim\limits_{x\to\infty} x^3 e^{-x^2} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} = \dfrac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} \overset{H}{=} \lim_{x\to\infty} \frac{-x^2}{2xe^{x^2}} = \lim_{x\to\infty} \frac{-x}{2e^{x^2}} = \frac{-\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{-x}{2e^{x^2}} \overset{H}{=} \lim_{x\to\infty} \frac{-1}{4xe^{x^2}} = 0$$

- $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right) = \lim\limits_{x\to 1} \left(\dfrac{x-1-\ln x}{\ln x(x-1)}\right) = \dfrac{0}{0}$

$$\lim_{x\to 1} \left(\frac{x-1-\ln x}{\ln x(x-1)}\right) \overset{H}{=} \lim_{x\to 1} \frac{1-\frac{1}{x}}{(\ln x)(1) + (x-1)\frac{1}{x}} \bullet \frac{x}{x} = \lim_{x\to 1}\left(\frac{x-1}{x\ln x + x - 1}\right) = \frac{0}{0}$$

$$\lim_{x\to 1}\left(\frac{x-1}{x\ln x + x - 1}\right) \overset{H}{=} \lim_{x\to 1}\left(\frac{1}{x\left(\frac{1}{x}\right) + \ln x + 1}\right) = \lim_{x\to 1} \frac{1}{\ln x + 2} = \frac{1}{2}$$