ANTIDERIVATIVES

Objective: Find a function whose derivative is a known function

You are no doubt familiar with the game show "Jeopardy." Alex reads the answer and you have to answer with the question. Let's play "Calculus Jeopardy."

- Answer: "A common function has a derivative of $f'(x) = 2x$."
  - What do you think the question was? How about, "What is f(x) = x^2. "
  - How would you justify your answer? "Based on the power rule, if f(x) = x^2, then $f'(x) = 2x$."
  - Alex grimaces and stalls for time. "More specific," he says.
  - You try again, "What is f(x) = $x^2 - 6$? After all, based on the power rule, if f(x) = $x^2 - 6$? then $f'(x) = 2x$."
  - "Sorry," says Alex.
  - Another player buzzes in and says, "What is f(x) = $x^2 + C$? "
  - "Correct," Alex says.

Where did you go wrong? The problem is that there are an infinite number of antiderivatives for any given function. The reason for this multiplicity of antiderivatives lies in the fact that the derivative of any constant term is zero.

- Suppose that $f'(x) = 3x^2$
  - Then f(x) = x^3, f(x) = $x^3 + \pi$, f(x) = $x^3 - 4e^5$, and f(x) = x^3 + C are all antiderivatives because in each case $f'(x) = 3x^2$.
  - When you are asked to find an antiderivative, you must include "+ C" (where C is an arbitray real constant). This covers all possibilities, even if C happens to equal zero.
  - The term "C" will be called the constant of integration in the next chapter. Each time you forget it on a test item, it will cost you you two points!
  - Note that you can check your answer by taking its derivative. This should get you back to the original function. You **MUST KNOW** the derivatives of all elementary functions taught in Calc I!

Let's examine some more polynomial functions and their antiderivatives

- If $f'(x) = 2x$, then $f(x) = x^2 + C$ because $\frac{dy}{dx}(x^2 + C) = 2x$
- If $f'(x) = 3x^2$, then $f(x) = x^3 + C$ because $\frac{dy}{dx}(x^3 + C) = 3x^2$
- If $f'(x) = 4x^3$, then $f(x) = x^4 + C$ because $\frac{dy}{dx}(x^4 + C) = 4x^3$
- Do you see a pattern here???

The power rule: If $f'(x) = x^n$ and $n \neq -1$, then $f(x) = \frac{1}{n+1}x^{n+1} + C$ is the most general antiderivative because $f'(x) = (n+1)\left(\frac{1}{n+1}x^{(n+1)-1}\right) = x^n = f(x)$. Why can't n = – 1??

- Your text may name the functions differently; their version of the above problem is:
  - If $f(x) = x^n$ $(n \neq -1)$, then $F(x) = \frac{1}{n+1}x^{n+1} + C$ is the most general antiderivative because $F'(x) = (n+1)\left(\frac{1}{n+1}x^{(n+1)-1}\right) = x^n = f(x)$.

Summary of antidifferentiation formulas which you **MUST KNOW**

Function	The most general antiderivative	One particular antiderivative
$f'(x) = x^n,\ n \neq -1$	$f(x) = \frac{x^{n+1}}{n+1} + C$	$f(x) = \frac{x^{n+1}}{n+1} + 8$
$f'(x) = \frac{1}{x}$	f(x) = ln | x | + C	f(x) = ln | x |
$f'(x) = e^x$	$f(x) = e^x + C$	$f(x) = e^x - \ln 4$
$f'(x) = \cos x$	f(x) = sin x + C	$f(x) = \sin x + \sqrt{\pi}$
$f'(x) = \sin x$	f(x) = – cos x + C	f(x) = – cos x – 3.7
$\sec^2 x\ f'(x) =$	f(x) = tan x + C	$f(x) = \tan x + \frac{2}{9}$
$f'(x) = \frac{1}{\sqrt{1-x^2}}$	$f(x) = \sin^{-1}x + C$	$f(x) = \sin^{-1}x$
$f'(x) = \frac{1}{1+x^2}$	$f(x) = \tan^{-1}x + C$	$f(x) = \tan^{-1}x + \pi - e$
$g'(x) = c\,f'(x)$	g(x) = cf(x) + C	g(x) = cf(x) + 5
$h'(x) = f'(x) + g'(x)$	h(x) = f(x) + g(x) + C	h(x) = f(x) + g(x) + 2006

Find the (general) antiderivative of each of the following

- $f(x) = 2 + \sqrt[5]{x^3}$

- $f(x) = 5\sec^2 x + 7x^2 + 4x^{3/5} - \frac{e}{x} + 1$

- Remember to check your work by taking the derivative of your answer; this should get you back to the original problem!

Some problems contain **initial conditions** which make it possible (and necessary!) for us to evaluate the constant term. Any equation that involves one or more derivatives of a function is called a **differential equation**.

- Find f(x) if $f'(x) = e^x + \frac{20}{1+x^2}$ and $f(0) = -2$
  - $f(x) = e^x + 20\tan^{-1}x + C$ is the ***general solution***.
  - $f(0) = e^0 + 20\tan^{-1}0 + C = -2 \Rightarrow C = -2 - 1 - 0 = -3$
  - Therefore, $f(x) = e^x + 20\tan^{-1}x - 3$ is the ***particular solution.***

- Find f(x) if $f''(x) = 12x^2 + 6x - 4$, if $f(0) = 4$ and $f(1) = 1$
  - $f'(x) = 12\cdot\frac{x^3}{3} + 6\cdot\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$
  - $f(x) = 4\cdot\frac{x^4}{4} + 3\cdot\frac{x^3}{3} - 4\cdot\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$
    - $f(0) = 0 + 0 - 0 + 0 + D = 4 \Rightarrow D = 4$
    - $f(1) = 1 + 1 - 2 + C + 4 = 1 \Rightarrow C = -3$
    - Therefore, $f(x) = x^4 + x^3 - 2x^2 + 4x - 3$

Find f(x): $f''(x) = 12x^2 - 6x + 2,\ f(0) = 1,\ f(2) = 11$

Rectilinear motion

- If s = f(t) is the position function of a particle, then the velocity function is v(t) = $s'(t)$; i.e. the the position function is an antiderivative of the velocity function!
- If v = v(t) is the velocity function, then the acceleration function is a(t) = $v'(t)$; i.e. the velocity function is an antiderivative of the acceleration function.
- Example: A ball is thrown upward with a speed of 24 ft/s from a height of 432 ft.
  - Find its height above the ground t seconds later.
    - Motion is vertical; choose upward to be the positive direction.
    - The only acceleration is the force of gravity, which is pulling downward.
      - a(t) = $\frac{dv}{dt} = -32\ ft/s^2$
      - v(t) = $-32t + C$ and $v_0 = 24 \Rightarrow C = 24$
      - The velocity function is v(t) = $-32t + 24$
      - s(t) = $-16t^2 + 24t + D$ and $s_0 = 432 \Rightarrow$ D = 432
      - The position function is $s(t) = -16t^2 + 24t + 432$
  - When does the ball reach its maximum height? What is its maximum height?
    - Maximum height $\Rightarrow$ v(t) = 0
    - v(t) = – 32t + 24 = 0 $\Rightarrow$ t = 0.75 s
    - The ball reaches its maximum height after 0.75 s.
    - Its maximum height is s(0.75) = 441 ft
  - When does the particle hit the ground?
    - Impact $\Rightarrow s(t) = 0$
    - s(t) = $-16t^2 + 24t + 432 = 0 \Rightarrow -16t^2 + 24t + 432 = 0$, so $t = \frac{3 \pm 21}{4} = -4.5, 6$

      The negative solution does not apply to this problem, so the ball hits the ground after 6 s.
  - What is the ball's impact velocity?
    - v(0.75) = -168
    - The ball's impact velocity is 168 ft/s.

A ball is dropped from the edge of a cliff which is 432 ft above the ground. Find the acceleration, velocity and position functions. Determine when the ball will hit the ground and find its impact velocity.

solutions:

- $f(x) = 2 + \sqrt[5]{x^3} = 2 + x^{3/5} \Rightarrow F(x) = 2x + \frac{5}{8}x^{8/5} + C$

- $f(x) = 5\sec^2 x + 7x^2 + 4x^{3/5} - \frac{e}{x} + 1 \Rightarrow$

  $F(x) = 5\tan x + \frac{1}{3}\left(7x^3\right) + \frac{5}{8}\left(4x^{8/5}\right) - e\ln\left|x\right| + x + C =$

  $F(x) = 5\tan x + \frac{7}{3}x^3 + \frac{5}{2}x^{8/5} - e\ln\left|x\right| + x + C$

- $f''(x) = 12x^2 - 6x + 2,\ f(0) = 1,\ f(2) = 11$

  $f'(x) = \frac{1}{3} * 12x^3 - \frac{1}{2} * 6x^2 + 2x + C = 4x^3 - 3x^2 + 2x + C$

  $f(x) = \frac{1}{4} * 4x^4 - \frac{1}{3} * 3x^3 + \frac{1}{2} * 2x^2 + Cx + D = x^4 - x^3 + x^2 + Cx + D$

  $f(0) = 0 - 0 + 0 + 0 + D = 1 \Rightarrow D = 1$

  $f(2) = 16 - 8 + 4 + 2C + 1 = 11 \Rightarrow C = -1$

  $f(x) = x^4 - x^3 + x^2 - x + 1$

- The only acceleration is the force of gravity, which is pulling downward.

  $a(t) = \dfrac{dv}{dt} = -32\ ft/s^2$

  $v(t) = -32t + C$ and $v_0 = 0 \Rightarrow C = 0$

  $v(t) = -32t$ ft/s

  $s(t) = -16t^2 + D$ and $s_0 = 432 \Rightarrow D = 432$

  $s(t) = -16t^2 + 432$ ft

  The maximum height is 432 ft because it is a free fall problem.

  When the ball hits the ground s(t) = 0, so $-16t^2 + 432 = 0 \Rightarrow t = \pm\sqrt{27} \approx \pm 5.20$

  The ball hits the ground after $3\sqrt{3}$ s. Its impact velocity is $96\sqrt{3} \approx 166.28$ ft/s.