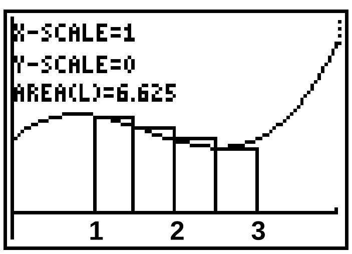
AREAS AND DISTANCES

Objectives: Find the area under a curve and the total distance traveled by a car

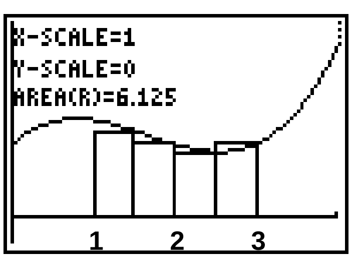
The area between the graph of a function and the x-axis on a closed interval can be estimated by drawing rectangles which approximate the area on that interval. We will use a relatively small number of rectangles to illustrate the process, but we could get surprisingly accurate results by increasing the number of rectangles.

Example: Estimate the area under the graph of $f(x) = .5x^3 - 2.5x^2 + 3x + 3$ (and above the x-axis) from x = 1 to x = 3 using 4 rectangles.

- Although the rectangles do not all have to have the same width, it will greatly simplify our calculations if they do!
- Start with a reasonably accurate graph of the function on the given interval.
- Draw the rectangles
- To find the width of each rectangle we will divide the given interval into subintervals
  - The width of each subinterval = $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

- There are three different methods that are used to find the height of the rectangles.
  - **Using left endpoints**: the height of each rectangle equals the value of the function at the left endpoint of the subinterval.
  - 

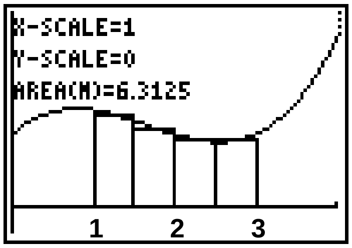
$$A \approx L_4 = \Delta x \bullet f(1) + \Delta x \bullet f(1.5) + \Delta x \bullet f(2) + \Delta x \bullet f(2.5)$$
$$= \frac{1}{2}\left(4 + \frac{57}{16} + 3 + \frac{43}{16}\right) = 6.625$$

    - From the graph we can see that this is an overestimate.
  - **Using right endpoints**: the height of each rectangle equals the value of the function at the right endpoint of the subinterval.
  - 

$$A \approx R_4 = \Delta x \bullet f(1.5) + \Delta x \bullet f(2) + \Delta x \bullet f(2.5) + \Delta x \bullet f(3)$$
$$= \frac{1}{2}\left(\frac{57}{16} + 3 + \frac{43}{16} + 3\right) = 6.125$$

    - From the graph we can see that this is an underestimate.

- **Using midpoints**: the height of each rectangle equals the value of the function at the midpoint of the subinterval.

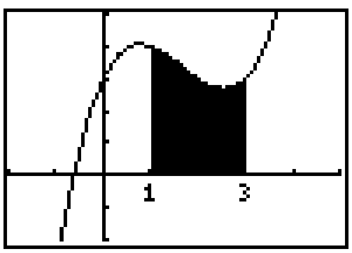
- 

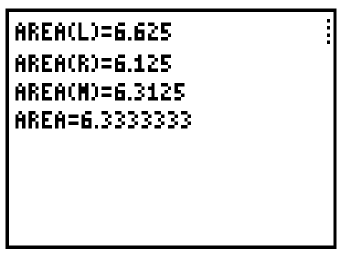
  - From the graph we can see that this is a good estimate.

$$A \approx M_4 = \Delta x \bullet f(1.25) + \Delta x \bullet f(1.75) + \Delta x \bullet f(2.25) + \Delta x \bullet f(2.75)$$
$$= \frac{1}{2}\left(\frac{489}{128} + \frac{419}{128} + \frac{357}{128} + \frac{351}{128}\right) = 6.3125$$

- The exact area can be found using an integral

  - The midpoint approximation is the most accurate approximation





- The estimate can be improved by using a larger value for n.

Find the area under the graph of $y = x^2 + 1$ from x = – 1 to x = 3 using 4 rectangles and a) right endpoints, b) left endpoints, and c) midpoints. From your graphs, which approximation appears best?

Example: Find the **exact area** under the graph of $y = x^2$ on the interval [0, 1].

- We will use n subintervals; then $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$
- Using right endpoints: $R_n = \dfrac{1}{n}\left(\dfrac{1}{n}\right)^2 + \dfrac{1}{n}\left(\dfrac{2}{n}\right)^2 + \dfrac{1}{n}\left(\dfrac{3}{n}\right)^2 + \dfrac{1}{n}\left(\dfrac{4}{n}\right)^2 + \ldots + \dfrac{1}{n}\left(\dfrac{n}{n}\right)^2$

$$= R_n = \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2\right)$$

NOTE: This is actually a very simple problem because the left endpoint of the interval is 0. If the interval were [4, 7], then the first endpoint $\left(\dfrac{1}{n}\right)$ would be $\left(4 + \dfrac{1}{n}\right)$ and the next endpoint would be $\left(4 + \dfrac{2}{n}\right)$, etc.

HINT: Using right endpoints is a little easier than left endpoints. You know what the last term is because the $\left(\dfrac{n}{n}\right) = 1$, which is the right endpoint of the interval.

- We can use a summation formula to get an expression for this sum.
  - $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

    "The sum of the squares of the integers from 1 to n is equal to ..."
  - The area is equal to the limit of this sum as n becomes infinitely large.
  - $A = \displaystyle\lim_{n\to\infty} \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \lim_{n\to\infty} \frac{1}{6}\left(\frac{n(n+1)(2n+1)}{n^3}\right)$

    $= \displaystyle\frac{1}{6}\lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}(1)(2) = \frac{1}{3}$
  - The exact area under the curve is $\dfrac{1}{3}$ square units.

Find the exact area under the graph of $y = x^2$ on the interval [0, 2].

Distance traveled: If the odometer is broken on your car, how could you estimate how far you have driven? You might have your passenger record the velocity (speedometer reading) every 5 seconds to generate a table as follows:

Time in s	0	5	10	15	20	25	30
Velocity in mph	17	21	24	29	32	31	28
Velocity in ft/s	25	31	35	43	47	45	41

- Since time is given in seconds, we must convert velocity to ft/s
- Use the factor-label method to convert mph to ft/s; round to nearest whole number
- $17\frac{\cancel{mi}}{1\cancel{hr}} \bullet \frac{1\cancel{hr}}{\cancel{60min}} \bullet \frac{\cancel{1min}}{60s} \bullet \frac{5280\,ft}{1\cancel{mi}} = 24.93 \approx 25\ ft/s$

Use the factor-label method to convert the rest of the velocities to ft/s and complete the table.

- Each time period is 5 s; let's use the velocity at the beginning of each time period

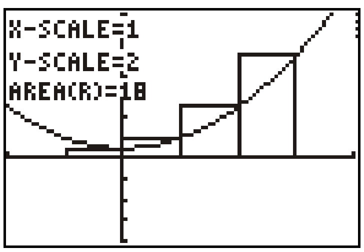
- D = rt $\Rightarrow$ distance traveled = rt = $25 \cdot 5 + 31 \cdot 5 + 35 \cdot 5 + 43 \cdot 5 + 47 \cdot 5 + 45 \cdot 5 =$
  = 1135 ft
- The car has traveled approximately 1135 ft during the first 30 s.

Estimate the distance traveled using the velocity at the end of each time period

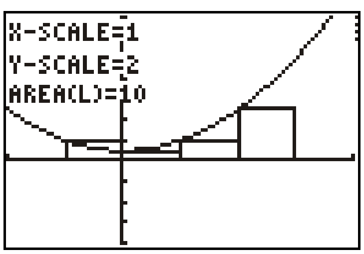
Solutions

- Area under the graph of $y = x^2 + 1$ on [–1, 3] using 4 rectangles

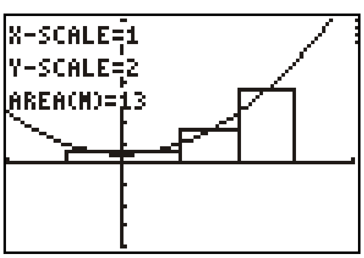
$$\Delta x = \frac{b - a}{n} = \frac{3 + 1}{4} = 1$$



$$A \approx R_4 = \Delta x \cdot f(0) + \Delta x \cdot f(1) + \Delta x \cdot f(2) + \Delta x \cdot f(3)$$
$$= \Delta x\,(f(0) + f(1) + f(2) + f(3))$$
= 1(1 + 2 + 5 + 10) = 18



$$A \approx L_4 = \Delta x \cdot f(-1) + \Delta x \cdot f(0) + \Delta x \cdot f(1) + \Delta x \cdot f(2)$$
$$= \Delta x\,(f(-1) + f(0) + f(1) + f(2))$$
= 1(2 + 1 + 2 + 5) = 10



$$A \approx M_4 = \Delta x \cdot f(-0.5) + \Delta x \cdot f(0.5) + \Delta x \cdot f(1.5) + \Delta x \cdot f(2.5)$$
$$= \Delta x\,(f(-0.5) + f(0.5) + f(1.5) + f(2.5))$$
= 1(1.25 + 1.25 + 3.25 + 7.25) = 13

This is probably the best approximation.

- The exact area under the graph of $y = x^2$ on the interval [0, 2].

  Use n subintervals; then $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

  Using right endpoints: $R_n = \frac{2}{n}\left(\frac{2}{n}\right)^2 + \frac{2}{n}\left(\frac{4}{n}\right)^2 + \frac{2}{n}\left(\frac{6}{n}\right)^2 + \frac{2}{n}\left(\frac{8}{n}\right)^2 + \ldots + \frac{2}{n}\left(\frac{2n}{n}\right)^2$

  NOTE: The last quantity $\left(\frac{2n}{n}\right) = 2$ which is the right endpoint of the interval.

  $$R_n = \frac{2}{n^3}(2^2)\left(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2\right) = \frac{8}{n^3}\left(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2\right)$$

  HINT: Factoring out the 2^2 leaves you with a familiar quantity!

  $$A = \lim_{n\to\infty}\left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty}\left(\frac{4}{3}\right)\frac{n(n+1)(2n+1)}{n^3}$$

  $$= \left(\frac{4}{3}\right)\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \left(\frac{4}{3}\right)\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \left(\frac{4}{3}\right)(1)(2) = \frac{8}{3}$$

  The exact area is $\frac{8}{3}$ square units.

- Velocities converted from mph to ft/s

$$21\frac{\text{mi}}{1\text{ hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}\cdot\frac{5280\text{ ft}}{1\text{ mi}} \approx 31\text{ ft/s} \qquad 32\frac{\text{mi}}{1\text{ hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}\cdot\frac{5280\text{ ft}}{1\text{ mi}} \approx 47\text{ ft/s}$$

$$24\frac{\text{mi}}{1\text{ hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}\cdot\frac{5280\text{ ft}}{1\text{ mi}} = 35\text{ ft/s} \qquad 31\frac{\text{mi}}{1\text{ hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}\cdot\frac{5280\text{ ft}}{1\text{ mi}} = 45\text{ ft/s}$$

$$29\frac{\text{mi}}{1\text{ hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}\cdot\frac{5280\text{ ft}}{1\text{ mi}} = 43\text{ ft/s} \qquad 28\frac{\text{mi}}{1\text{ hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}\cdot\frac{5280\text{ ft}}{1\text{ mi}} \approx 41\text{ ft/s}$$

- Using right endpoints:
  D = rt ⇒ distance traveled = rt = $31\cdot5 + 35\cdot5 + 43\cdot5 + 47\cdot5 + 45\cdot5 + 41\cdot5 =$
  = 1210 ft
- The car has traveled approximately 1210 ft during the first 30 s.