INTEGRATION OF IMPROPER INTEGRALS

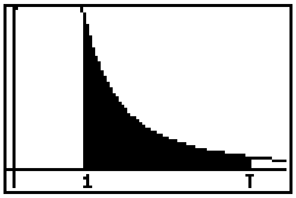
Objectives: Evaluate integrals on infinite intervals and intervals which have infinite discontinuities

Improper integrals

- May be defined on an infinite interval
- May have an infinite discontinuity
- Are used in probability distributions

The improper integrals $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{b} f(x)dx$ are said to be convergent if the corresponding [finite] limit exists and divergent if the [finite] limit does not exist.

Type I: Infinite intervals



- $A(t) = \int_1^t \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}$
- The unbounded region extends indefinitely in a horizontal direction
- Note that $A(t) < 1$ no matter how large t is chosen
- $\lim\limits_{t \to \infty} A(t) = \lim\limits_{t \to \infty}\left(1 - \frac{1}{t}\right) = 1$
- Area of shaded region approaches 1 as $t \to \infty$
- $\int_1^{\infty} \frac{1}{x^2}dx = \lim\limits_{t \to \infty}\int_1^t \frac{1}{x^2}dx = 1$
- The integral is convergent

Integrate $\int_{-\infty}^{0} \frac{1}{2x-5}dx$

HINT: $\int_{-\infty}^{0} \frac{1}{2x-5}dx = \lim\limits_{t \to \infty}\int_t^0 \frac{1}{2x-5}dx$

Type 1 improper integral

- If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$, provided this [finite] limit exists
- If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$, provided this [finite] limit exists
- The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are said to be convergent if the corresponding [finite] limit exists and divergent if the [finite] limit does not exist.
- If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

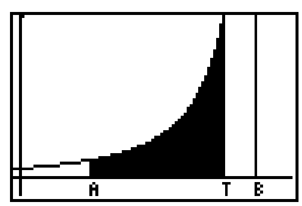
  $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$ for any real number a.

Is $\int_1^\infty \frac{1}{x}dx$ convergent or divergent?

- $\lim_{t \to \infty} \int_1^t \frac{1}{x}dx = \lim_{t \to \infty} [\ln|x|]_1^t = \lim_{t \to \infty} (\ln t - \ln 1) = \lim_{t \to \infty} (\ln t) = \infty$
- Therefore, $\int_1^\infty \frac{1}{x}dx$ is divergent.
- Compare this result to $\int_1^\infty \frac{1}{x^2}dx$ which converges
- In general, $\int_1^\infty \frac{1}{x^p}dx$ is convergent if p **>** 1 and divergent if $p \leq 1$

Type 2: Discontinuous integrands

- f is continuous on a finite interval [a, b)
- The unbounded region is infinite in a vertical direction



Type 2 improper integrals

- If f is continuous on [a, b) and is discontinuous at b, then $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$, if this [finite] limit exists
- If f is continuous on [a, b) and is discontinuous at a, then $\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$, if this [finite] limit exists

- The improper integral $\int_a^b f(x)dx$ is said to be convergent if the corresponding [finite] limit exists and divergent if the [finite] limit does not exist
- If f has a discontinuity at c, where a **<** c **<** b, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Evaluate $\int_0^3 \frac{dx}{x-1}$

- This is not an ordinary definite integral because there is a vertical asymptote x = 1
- Improper integrals must be calculated in terms of limits!
- $\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$,

  where $\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} [\ln|x-1|]_0^t = \lim_{t\to 1^-} (\ln|t-1| - \ln|-1|)$

  $= \lim_{t\to 1^-} \ln(1-t) = -\infty$
- Since $\int_0^1 \frac{dx}{x-1}$ is divergent, so is $\int_0^3 \frac{dx}{x-1}$

Comparison test for improper integrals

- Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$
- If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is also convergent
- If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is also divergent

Solutions

- $\int_{-\infty}^0 \frac{1}{2x-5}dx = \lim_{t\to\infty} \int_t^0 \frac{1}{2x-5}dx = \lim_{t\to-\infty} \left[\frac{1}{2}\ln|2x-5|\right]_t^9 = \lim_{t\to-\infty} \left[\frac{1}{2}\ln 5 - \frac{1}{2}\ln|2t-5|\right] = -\infty$

  divergent