THE DEFINITE INTEGRAL

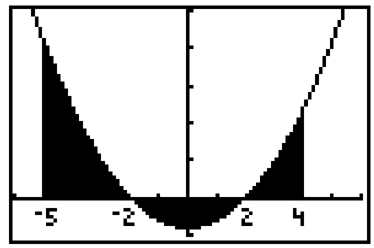
Objective: Evaluate definite integrals using Riemann sums

If f is **continuous** on [a, b], then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a ***Riemann sum***

Given: $f(x) = x^2 - 4$



- Use the Midpoint Rule with $\Delta x = \frac{1}{2}$ to approximate the shaded area on [–5, 4].
- Find the zeros of the function to separate the shaded area into 3 regions. Each region must be entirely above the x-axis or entirely below.
- $A_1 \approx M_6 = \frac{1}{2}f(-4.75) + \frac{1}{2}f(-4.25) + \frac{1}{2}(-3.75) + \frac{1}{2}(-3.25) + \frac{1}{2}(-2.75) + \frac{1}{2}(-2.25)$
  = 26.9375

Find $A_2 \approx M_8$ **HINT**: Your result should be negative

Find $A_3 \approx M_4$

- If we are approximating area, we must use the absolute value of any “areas” where the calculations result in a negative number.
  - Total area $\approx$ 26.9375 + |-10.75| + 10.625 = 48.3125
- If we are approximating the integral, then the signs are kept intact.
  - $\int_{-5}^{4}(x^2 - 4)dx \approx$ 26.9375 – 10.75 + 10.625 = 26.8125 ***Riemann sum***

How is the integral related to the shaded areas?

Use the limit of the Riemann sum to find the exact value of $\int_{-5}^{4} (x^2 - 4)dx$

- $\Delta x = \dfrac{b-a}{n} = \dfrac{9}{n}$
- $x_0 = -5,\ x_1 = -5 + \dfrac{9}{n},\ x_2 = -5 + \dfrac{18}{n},\ x_3 = -5 + \dfrac{27}{n}$, and, in general, $x_i = -5 + \dfrac{9i}{n}$.
- $\int_{-5}^{4} (x^2 - 4)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f\left(-5 + \dfrac{9i}{n}\right)\dfrac{9}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9}{n}\left[\left(-5+\frac{9i}{n}\right)^2 - 4\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9}{n}\left[25 - \frac{90i}{n} + \frac{81i^2}{n^2} - 4\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9}{n}\left[21 - \frac{90i}{n} + \frac{81i^2}{n^2}\right] = \lim_{n\to\infty} \sum_{i=1}^{n}\left[\frac{189}{n} - \frac{810i}{n^2} + \frac{729i^2}{n^3}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 189 - \frac{810}{n^2}\sum_{i=1}^{n} i + \frac{729}{n^3}\sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}(189n) - \frac{810}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{729}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty} \left[189 - 405\left(\frac{n(n+1)}{n^2}\right) + \frac{243}{2}\left(\frac{n(n+1)(2n+1)}{n^3}\right)\right]$$

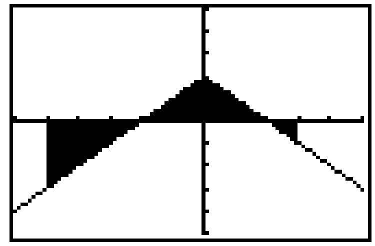
$$= \lim_{n\to\infty} \left[189 - 405\left(\frac{n+1}{n}\right) + \frac{243}{2}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[189 - 405\left(1+\frac{1}{n}\right) + \frac{243}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= 189 - 405(1) + \frac{243}{2}(1)(2) = 27$$

Use the limit of the Riemann sum to find the exact value of $\int_{0}^{3} (x^2 - 4)dx$

Evaluate $\int_{-5}^{3}(2-|x|)dx$ by interpreting it in terms of areas.



- First find the zeros of the function on the required interval; they are – 2 and 2.
- $\int_{-5}^{-2}(2-|x|)dx = -\left(\frac{1}{2}\cdot 3\cdot 3\right) = -\frac{9}{2}$
- $\int_{-2}^{2}(2-|x|)dx = \left(\frac{1}{2}\cdot 4\cdot 2\right) = 4$
- $\int_{2}^{3}(2-|x|)dx = -\left(\frac{1}{2}\cdot 1\cdot 1\right) = -\frac{1}{2}$
- $\int_{-5}^{3}(2-|x|)dx = \int_{-5}^{-2}(2-|x|)dx + \int_{-2}^{2}(2-|x|)dx + \int_{2}^{3}(2-|x|)dx = -\frac{9}{2} + 4 - \frac{1}{2} = -1$
- We can find the exact value of the integral because the shaded areas are triangles.

Important properties of the definite integral

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b c\,dx = c(b-a)$, where c is any constant
- $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
- If $f(x) \geq 0$ on [a, b], then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ on [a, b], then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ on [a, b], then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Solutions

- $A_2 \approx M_8 =$

$\frac{1}{2}f(-1.75) + \frac{1}{2}f(-1.25) + \frac{1}{2}(-0.75) + \frac{1}{2}(-0.25) + \frac{1}{2}(0.25) + \frac{1}{2}(0.75) + \frac{1}{2}(1.25) + \frac{1}{2}(1.75)$

$= \frac{1}{2}\left(-0.9375 - 2.4375 - 3.4375 - 3.9375 - 3.9375 - 3.4375 - 2.4370 - 0.9375\right)$
$= -10.75$

- $A_3 \approx M_4 = \frac{1}{2}f(2.25) + \frac{1}{2}f(2.75) + \frac{1}{2}(3.25) + \frac{1}{2}(3.75)$

  $= \frac{1}{2}(1.0625 + 3.5625 + 6.5625 + 10.0625) = 10.625$
- The integral is equal to the area of the shaded regions above the x-axis minus the area of the shaded regions below the x-axis.
- Exact value of $\int_0^3 (x^2 - 4)dx$
  - $\Delta x = \frac{b-a}{n} = \frac{3}{n}$
  - $x_0 = 0,\ x_1 = \frac{3}{n},\ x_2 = \frac{6}{n},\ x_3 = \frac{9}{n}$, and, in general, $x_i = \frac{3i}{n}$.
  - $\int_0^3 (x^2 - 4)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{3i}{n}\right)^2 - 4\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\frac{9i^2}{n^2} - 4\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\sum_{i=1}^{n}\left[i^2\right] - \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}[4]$$

$$= \lim_{n\to\infty}\left[\frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{3}{n}(4n)\right]$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\left(\frac{(n+1)(2n+1)}{n^2}\right) - 12\right]$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - 12\right]$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 12\right]$$

$$= \frac{9}{2}(1)(2) - 12 = -3$$