EVALUATING DEFINITE INTEGRALS

Objective: Evaluate definite integrals using the evaluation theorem

If f is ***continuous*** on [a, b], then the definite integral of f from x = a to

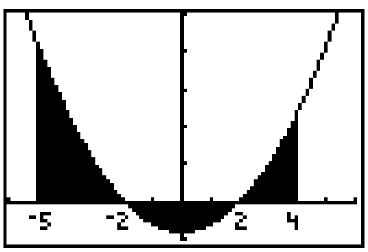
x = b is $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = F(b) - F(a)$, where F is ***any*** antiderivative of f.

Since F can be any antiderivative, we may choose to let C = 0! If you choose C to be any nonzero real number, it will just cancel out in the evaluation process.

Evaluation theorem: $\int_a^b f(x)\,dx = F(b) - F(a)$

Example: Find $\int_{-5}^{4} (x^2 - 4)\,dx$ and interpret the result in terms of areas.

$$\int_{-5}^{4} (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-5}^{4} = \left[\frac{4^3}{3} - 4(4)\right] - \left[\frac{(-5)^3}{3} - 4(-5)\right] = 27$$



1. $\int_{-5}^{-2} (x^2 - 4)\,dx \approx M_6 = 26.9375$

   $$\int_{-5}^{-2} (x^2 - 4)\,dx = \left[\frac{(-2)^3}{3} - 4(-2)\right] - \left[\frac{(-5)^3}{3} - 4(-5)\right] = 27$$

2. $\int_{-2}^{2} (x^2 - 4)\,dx \approx M_8 = -10.75$

   $$\int_{-2}^{2} (x^2 - 4)\,dx = \left[\frac{(2)^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right] = -\frac{32}{3}$$

3. $\int_{2}^{4} (x^2 - 4)\,dx \approx M_4 = 10.625$

   $$\int_{2}^{4} (x^2 - 4)\,dx = \left[\frac{(4)^3}{3} - 4(4)\right] - \left[\frac{(2)^3}{3} - 4(2)\right] = \frac{32}{3}$$

4. $\int_{-5}^{4}(x^2 - 4)\,dx \approx M_{18} = 26.8125$

$$\int_{-5}^{4}(x^2 - 4)\,dx = 27 - \frac{32}{3} + \frac{32}{3} = 27$$

The definite integral equals the sum of the areas above the x-axis minus the areas below the x-axis

Summary: To find the exact value of $\int_{0}^{4}(x^2 - 4)\,dx$

Apply equation 3 from Section 5.2 [p. 354], a lengthy tedious process

$$\int_{0}^{4}(x^2 - 4)\,dx = \frac{16}{3}$$

Use the evaluation theorem from Section 5.3 [p.366]

$$\int_{0}^{4}(x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_0^4 = \left[\frac{(4)^3}{3} - 4(4)\right] - \left[\frac{(0)^3}{3} - 4(0)\right] = \frac{16}{3}$$

Evaluate $\int_{1}^{4}\frac{1}{\sqrt{x}}\,dx$

Evaluate $\int_{\pi}^{2\pi}\cos\theta\,d\theta$

The ***indefinite*** integral $\int f(x)dx$ is a ***function***.

The ***indefinite*** integral $\int f(x)dx$ is the set of all ***antiderivatives*** of the function F(x), where $f'(x) = F(x)$.

Example: $\int \frac{1}{x}dx = \ln|x| + C$

C is called the ***constant of integration***.
C may be any real number, but can only be evaluated if an ***initial condition*** is given

The ***definite*** integral $\int_a^b f(x)\,dx$ is a ***number***.

Example: $\int_1^e \frac{1}{x}dx = \Big[\ln|x|\Big]_1^e = \ln(e) - \ln(1) = 1$

KNOW the table of indefinite integrals on page 369

These formulas are only valid on closed intervals where the function is ***<u>continuous</u>***

$$\int f(x)\,dx = F(x) \Leftrightarrow F'(x) = f(x)$$

<u>Total change theorem</u>

If y = F(x), then $F'(x)$ represents the ***rate of change*** of F(x) with respect to x, and F(b) – F(a) is the ***total change*** in y when x changes from a to b

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

The (ever-popular) moving particle problem: A particle moves along a line with velocity of $v(t) = (t^2 - t - 6)$ m/s at time t.

Find the displacement of the particle from t = 1 s to t = 4 s

Displacement = change in location relative to starting point

$$s(4) - s(1) = \int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2} = -4.5\text{ m}$$

This means the particle is now located 4.5 m ***left*** of its starting point

Find the total distance traveled during this time period

Recall that speed = |v(t)| = |(t – 3)(t + 2)|

v(t) **<** 0 on (1, 3) and v(t) **>** 0 on (3, 4)

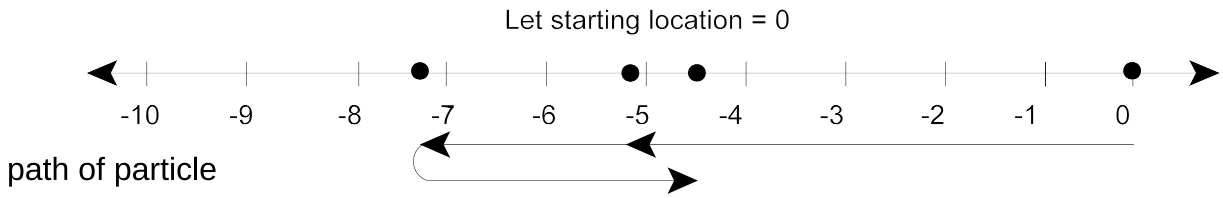
$$\text{Total distance traveled} = \int_1^4 |v(t)|dt = -\int_1^3 v(t)dt + \int_3^4 v(t)dt$$

$$= -\int_1^3 (t^2 - t - 6)dt + \int_3^4 (t^2 - t - 6)dt$$

$$= -\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 = \frac{61}{6} \approx 10.17\ \text{m}$$

Particle has moved a total of about 10.17 m during the period from t = 1 s to t = 4 s.

Visualization of the motion of the particle



time t in sec	3	2	4	1
displacement in m	-22/3	-31/6	-4.5	0
velocity in m/s	0	-4	6	-6

Find and interpret the displacement of the particle from t = 0 s to t = 5 s.

Find the total distance traveled during the first 5 s.

General motion of an object

If an object moves along a straight line with position function s(t) at time t,

Then its velocity is $v(t) = s'(t)$ at time t, (speed = | v(t) |)

And $\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$ is the ***total change of position***, or ***displacement***, of the particle during the period from time t_1 to time t_2.

If the velocity function of the object is v(t) at time t,

Then its ***acceleration*** (change in velocity) is $a(t) = v'(t)$ at time t,

And $\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1)$ is the ***total change*** in velocity from time t_1 to time t_2.

Some typical applications of the integral to find total change (see p. 371-2)

If V(t) = volume of water in a reservoir at time t,

Then $V'(t)$ is the ***rate*** at which water flows into (or out of) the reservoir at time t,

And $\int_{t_1}^{t_2} V'(t)dt = V(t_2) - V(t_1)$ is the ***total change*** in the amount of water in the reservoir between time t_1 and time t_2

If [C](t) is the concentration of the product C of a chemical reaction at time t

Then the ***rate of reaction*** is $\frac{d}{dt}[C]$ at time t,

And $\int_{t_1}^{t_2} \frac{d[C]}{dt} = [C](t_2) - [C](t_1)$ is the ***total change*** in the concentration of C from time t_1 to time t_2.

If the mass of a rod measured from the left end to point x is m(x),

Then the ***linear density*** (change in mass per unit length) is $\rho(x) = m'(x)$ at point x,

And $\int_a^b \rho(x)\,dx = m(b) - m(a)$ is the ***total mass*** of the segment of the rod that lies between x = a and x = b.

The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from the left end of the rod.

Find and interpret $\rho(3)$.

Find the [exact] total mass of the rod.

Find the [exact] mass of the first 3 m of the rod, measured from the left end.

Find the [exact] mass of the rod between 1 m and 2 m, measured from left end.

If n(t) represents a population at time t,

Then the ***rate of growth*** (or decay) of the population is $\frac{dn}{dt}$ at time t,

And $\int_{t_1}^{t_2} \frac{dn}{dt} = n(t_2) - n(t_1)$ is the ***total change*** in the population during the period period from time t_1 to time t_2.

If C(x) is the cost of producing x units of a commodity,

Then the ***marginal cost*** of producing the (x+1)st unit is $C'(x)$,

And $\int_{x_1}^{x_2} \frac{dC}{dt} = C(x_2) - C(x_1)$ is the total increase in cost when production is increased from x_1 units to x_2 units.

Power is the ***rate of change*** of energy: $P(t) = E'(t)$.

The unit of measurement for $\int_a^b f(x)\,dx$ is the product of the units for $f(x)$ and the units for x

Example: A particle moves along a line with velocity of $v(t) = (t^2 - t - 6)$ m/s at time t.

Displacement = $s(4) - s(1) = \int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)dt = -4.5$ m,

because v(t) is in m/s and dt is in s, therefore the product is in $\frac{m}{s}(s) = m$.

Solutions

$$\int_1^4 \frac{1}{\sqrt{x}}dx = \int_1^4 x^{-1/2}dx = \left[2x^{1/2}\right]_1^4 = 2(\sqrt{4} - \sqrt{1}) = 2$$

$$\int_\pi^{2\pi} \cos\theta d\theta = [\sin\theta]_\pi^{2\pi} = \sin 2\pi - \sin\pi = 0 - 0 = 0$$

Displacement: $s(5) - s(0) = \int_0^5 v(t)dt = \int_0^5 (t^2 - t - 6)dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_0^5 = -\frac{5}{6}$ m

After 5 s the particle is $\frac{5}{6}$ m to the left of its position at t = 0 s.

Total distance traveled: $\int_0^5 |v(t)|dt = -\int_0^3 v(t)dt + \int_3^5 v(t)dt$

$$= -\int_0^3 (t^2 - t - 6)dt + \int_3^5 (t^2 - t - 6)dt$$

$$= -\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_0^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^5 = 13.5 + \frac{38}{3} = \frac{157}{6} \text{ m} \approx 26.17 \text{ m}$$

Particle has moved a total of about 26.17 m during the period from t = 0 s to t = 5 s.

Linear density at x = 3 m = $\rho(3) = 9 + 2\sqrt{3}$ kg/m. The mass of the rod is changing at a rate of $9 + 2\sqrt{3}$ kilograms per meter at the point which is 3 m from the left end of the rod.

Total mass = m(4) – m(0) = $\int_0^4 (9 + 2\sqrt{x})dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = \frac{140}{3} - 0 = \frac{140}{3}$ kg

Mass of first 3 m from left end = m(3) – m(0) = $\int_0^3 (9 + 2\sqrt{x})dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^3$

$$= 27 + \frac{4}{3}\left(3^{3/2}\right) - 0 = 27 + 4\sqrt{3} \text{ kg}$$

Mass between 1m and 2 m = m(2) – m(1) = $\int_1^2 (9 + 2\sqrt{x})dx = \left[9x + \frac{4}{3}x^{3/2}\right]_1^2$

$$= 18 + \frac{4}{3}\left(2^{3/2}\right) - \left[9 + \frac{4}{3}\right] = 18 + \frac{8}{3}\sqrt{2} - \frac{31}{3} = \frac{23}{3} + \frac{8}{3}\sqrt{2} \text{ kg}$$