FUNDAMENTAL THEOREM OF CALCULUS

Objective: Investigate the link between differentiation and integration as given by the Fundamental Theorem of Calculus

Two properties of integrals

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- If a = b, then $\Delta x = 0$, so $\int_a^a f(x)\,dx = 0$

Comparison properties of integrals

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

  This can be interpreted as the area under the graph of a function where the area is entirely ***above the x-axis***.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

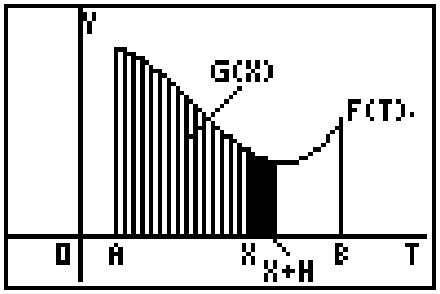
  This means that a "bigger" function has a "bigger" integral.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

  This means that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M

Motivation for the fundamental theorem: let f be a continuous function on [a, b] and define a function g by $g(x) = \int_a^x f(t)\,dt$ where $a \leq x \leq b$.

- g depends only on x
  - If x is a fixed number, then $\int_a^x f(t)\,dt$ is a definite number.
  - If x is a variable, then $\int_a^x f(t)\,dt$ also varies and defines a function of x, denoted by g(x)
- Example: Suppose $f(t) = t^2$ and a = 1
  - $g(x) = \int_1^x t^2\,dt = \dfrac{x^3 - 1}{3}$
  - $g'(x) = x^2$, that is $g'(x) = f(x)$
  -
- g is defined as the integral of f, then g turns out to be an antiderivative of f.

In general, consider any continuous function f with $f(x) \geq 0$.



- $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b.
- For $h > 0$, $g(x + h) - g(x)$ is the area under the graph of f from x to $(x + h)$.
  - If h is small, this area is approximately equal to the area of the rectangle with height f(x) and width h, that is $g(x + h) - g(x) \approx h[f(x)]$.
  - Then $\dfrac{g(x+h) - g(x)}{h} \approx f(x)$
- It appears that $g'(x) = \lim\limits_{h \to 0} \dfrac{g(x+h) - g(x)}{h} = f(x)$!

The fundamental theorem of calculus: Suppose f is continuous on [a, b]

Part 1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$

If f is integrated and the result is then differentiated, we arrive back at the original function f.

Part 2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

If we differentiate a function F, and then integrate the result, we arrive back at the original function F in the form of F(b) – F(a).

OPTIONAL: Proof of FTCI: Let $g(x) = \int_a^x f(t)\,dt$

A. If x and (x + h) are in the open interval (a, b), then

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt = \int_a^{x+h} f(t)\,dt$$

B. Then, for $h \neq 0$, $\dfrac{g(x+h) - g(x)}{h} = \dfrac{1}{h}\int_a^{x+h} f(t)\,dt$

C. Assume that $h > 0$.

1. Since f is continuous on [x, x + h], there are numbers u and v in [x, x + h] such that f(u) = m and f(v) = M, where m and M are the absolute minimum and maximum

values of f on [x, x + h].

2. Applying property 3 with b – a = (x + h) – x = h

$mh \le \int_a^{x+h} f(t)\,dt \le Mh$, that is, $f(u)h \le \int_a^{x+h} f(t)\,dt \le f(v)h$

$h > 0 \Rightarrow \; f(u) \le \dfrac{1}{h}\int_a^{x+h} f(t)\,dt \le f(v)$

3. From above: $f(u) \le \dfrac{g(x+h) - g(x)}{h} \le f(v)$
4. There is a similar argument for h < 0.

D. Let $h \to 0$

1. Then $u \to x$ and $v \to x$, since u and v lie between x and x + h.
2. Thus, $\lim\limits_{h\to 0} f(u) = \lim\limits_{u\to x} f(u) = f(x)$ and $\lim\limits_{h\to 0} f(v) = \lim\limits_{v\to x} f(v) = f(x)$ because f is continuous at x.
3. Applying the squeeze theorem (p. 114):

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = f(x)$$

OPTIONAL: Proof of FTCII: Divide the interval [a, b] into n subintervals with endpoints $x_0 = a,\ x_1,\ x_2,\ x_3,\ x_4,\ \ldots,\ x_n = b$ and with length $\Delta x = \dfrac{b-a}{n}$.

A. $F(b) - F(a) = F(x_n) - F(x_0)$

$= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_2) - F(x_1)] + [F(x_1) - F(x_0)]$

$= \sum\limits_{i=1}^{n} [F(x_i) - F(x_{i-1})]$

B. Applying the Mean Value Theorem (p. 279) to F on each subinterval $[x_i, x_{i-1}]$

1. There exists a number x_i^* between x_{i-1} and x_i such that

$F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = f(x_i^*)\Delta x$

2. Therefore, $F(b) - F(a) = \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

C. $\lim\limits_{n\to\infty}[F(b) - F(a)] = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\Delta x \to F(b) - F(a) = \int_a^b f(x)\,dx$

Examples using the Fundamental Theorem

- Find $g'(x)$ if $g(x) = \int_0^x (1+t^2)\,dt$

  - Using FTXII: $g(x) = \int_0^x (1+t^2)\,dt = \left[t + \frac{t^3}{3}\right]_0^x = x + \frac{x^3}{3} \Rightarrow g'(x) = 1 + x^2$

  - Using FTCI: Let $f(t) = 1 + t^2$; then $g(x) = \int_0^x (1+t^2)\,dt \Rightarrow g'(x) = f(x) = 1 + x^2$

- Find $g'(x)$ if $g(x) = \int_\pi^x \cos t\,dt$

  - Using FTXII: $g(x) = \int_\pi^x \cos t\,dt = [\sin t]_\pi^x = \sin x - \sin\pi \Rightarrow g'(x) = \cos x$

  - Using FTCI: Let $f(t) = \cos t$; then $g(x) = \int_\pi^x \cos t\,dt \Rightarrow g'(x) = f(x) = \cos x$

Find the derivative of each of the following functions.

$g(x) = \int_1^x \ln t\,dt$ $\qquad$ $F(x) = \int_x^{10} \tan\theta\,d\theta$

Solutions

Let $f(t) = \ln t$; then $g(x) = \int_1^x \ln t\,dt \Rightarrow g'(x) = f(x) = \ln x$

Let $f(\theta) = \tan\theta$; then $F(x) = \int_x^{10} \tan\theta\,d\theta = -\int_{10}^x \tan\theta\,d\theta \Rightarrow F'(x) = -f(x) = -\tan x$