

INTEGRATION BY SUBSTITUTION

Objective: Evaluate integrals using the substitution rule to change variables

The **substitution rule**

If $u = g(x)$ is a differentiable function whose range is an interval I , and f is continuous on I , then $\int f[g(x)]g'(x) dx = \int f(u) du$

Using substitution to find indefinite integrals: General steps

- Choose the **quantity** which “u” will represent. The choice is usually pretty obvious, but remember that you must be able to differentiate “u”: common choices include
 - Quantity which is raised to a power
 - Quantity which is the argument of a function
 - Higher power if multiple powers are involved
 - Exponent of “e”
 - $\ln x$
- Once you have chosen the quantity which will equal “u”, find the **differential** du:
 $u = f(x) \Rightarrow \frac{du}{dx} = f'(x) \Rightarrow du = f'(x) dx$ (p. 250)
- Rewrite the given integral in terms of u and du
- Substitute, using a substitution factor if necessary
- Integrate
- Back substitute
- Check by differentiating

Now that you have the plan, let's try to apply it!

Using substitution to find indefinite integrals

Example: Evaluate $\int 12x^2 \sqrt{4x^3 + 7} dx$

let $u = 4x^3 + 7$, then $du = 12x^2 dx$

$$\begin{aligned} \int 12x^2 \sqrt{4x^3 + 7} dx &= \int \underbrace{(4x^3 + 7)}_u \underbrace{^{1/2}}_{du} (12x^2 dx) \\ &= \int u^{1/2} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (4x^3 + 7)^{3/2} + C \end{aligned}$$

NOTE: It is not usually quite this easy! You frequently have to introduce a “fudge factor” to compensate for the fact that the derivative is not quite what you would like it to be. The fudge factor is a number, **not a variable!**

- Example: Evaluate $\int \sqrt[3]{(5x-7)^4} dx = \int \underbrace{(5x-7)}_u^{4/3} dx$ let $u = 5x + 7$, then $du = 5 dx$

But we do not have a factor of $(5dx)$. No problem, just write it down anyway.

$$\frac{1}{5} \int \underbrace{(5x-7)}_u^{4/3} \underbrace{(5 dx)}_{du}$$

A "fudge factor" of $\frac{1}{5}$ is used because $\frac{1}{5}(5dx) = dx$, which is what we had in the original problem. Since it is a number, it can be put in front of the integral symbol and we will worry about it later.

$$\text{So, we have } \frac{1}{5} \int (u)^{4/3} du = \frac{1}{5} \left(\frac{3}{7} u^{7/3} \right) + C = \frac{3}{35} (5x-7)^{7/3} + C$$



Evaluate $\int (8-4x)^5 dx$

- Example: Evaluate $\int 8 \cos(4x) dx$ let $u = 4x$, then $du = 4 dx$

$$\int 8 \cos(4x) dx = 2 \int \cos(\underbrace{4x}_u) (\underbrace{4 dx}_{du}) = 2 \int \cos(u) du = 2 \sin u + C = 2 \sin(4x) + C$$



Evaluate $\int -12 \csc(3x) \cot(3x) dx$

- Example: Evaluate $\int 2x^2(2x^3+1)^7 dx$ let $u = 2x^3 + 1$, then $du = 6x^2 dx$

$$\int 2x^2(2x^3+1)^7 dx = \frac{1}{3} \int \underbrace{(2x^3+1)}_u^7 \underbrace{(6x^2) dx}_{du} = \frac{1}{3} \int u^7 du = \frac{1}{3} \left(\frac{u^8}{8} \right) + C = \frac{(2x^3+1)^8}{24} + C$$



Evaluate $\int x^3 \sqrt{7x^4-5} dx$

- Example: Evaluate $\int \frac{7(x^2 - 1)}{(x^3 - 3x + 1)^6} dx$

let $u = x^3 - 3x + 1$, then $du = (3x^2 - 3)dx = 3(x^2 - 1)dx$

$$\int \frac{7(x^2 - 1)}{(x^3 - 3x + 1)^6} dx = \frac{7}{3} \int \frac{1}{\underbrace{(x^3 - 3x + 1)}_u^6} \underbrace{[3(x^2 - 1) dx]}_{du}$$

$$= \frac{7}{3} \int u^{-6} du = \frac{7}{3} \left(\frac{u^{-5}}{-5} \right) + C = \frac{-7}{15(x^3 - 3x + 1)^5} + C$$



Evaluate $\int e^{x^2+4x}(x+2)dx$

- Evaluate $\int [\cos^3(5t)] \sin(5t) dt$ let $u = \cos(5t)$, then $du = -5\sin(5t) dt$

$$\int [\cos^3(5t)] \sin(5t) dt = -\frac{1}{5} \int [\cos(5t)]^3 [-5 \sin(5t) dt] = -\frac{1}{5} \int u^3 du$$

$$= -\frac{1}{5} \left(\frac{u^4}{4} \right) + C = -\frac{1}{20} \cos^4(5t) + C$$

- Evaluate $\int \tan(x) dx = \int \frac{\sin(x)}{\cos(x)} dx$ let $u = \cos(x)$, then $du = -\sin(x) dx$

$$\int \frac{\sin(x)}{\cos(x)} dx = -\int \frac{du}{u} = -\ln |u| + C = -\ln |\cos(x)| + C$$

- Example: Evaluate $\int 3x(3x - 8)^{11} dx$ let $u = 3x - 8$, then $du = 3 dx$ and $x = \frac{u + 8}{3}$

In this problem we have a leftover variable. You cannot pull a variable out in front of the integral symbol, so we must replace it with some expression in terms of u .

$$\int 3x(3x - 8)^{11} dx = \int \frac{x}{\frac{u+8}{3}} \underbrace{(3x - 8)^{11}}_u \underbrace{(3 dx)}_{du} = \int \left(\frac{u + 8}{3} \right) u^{11} du = \int \left(\frac{u^{12}}{3} + \frac{8u^{11}}{3} \right) du$$

$$= \frac{1}{3} \left(\frac{u^{13}}{13} \right) + \frac{1}{3} \left(\frac{8u^{12}}{12} \right) + C = \frac{(3x - 8)^{13}}{39} + \frac{2(3x - 8)^{12}}{9} + C$$



Evaluate and check: $\int 4x(2x + 9)^8 dx$

Using substitution to find definite integrals:

Example: Evaluate $\int_{\pi/6}^{\pi/4} \cos^3(x) dx = \int_{\pi/6}^{\pi/4} [\cos^2(x)] \cos(x) dx = \int_{\pi/6}^{\pi/4} [1 - \sin^2(x)][\cos(x) dx]$

Note the trig substitution.

Method 1: **Omit limits of integration** until integration and back substitution are completed. This is the method I usually use.

Let $u = \sin x$, then $du = \cos(x) dx$

$$\int (1 - u^2) du = u - \frac{u^3}{3} = \sin(x) - \frac{\sin^3 x}{3}$$

Evaluate using original limits of integration:

$$\begin{aligned} \left[\sin(x) - \frac{\sin^3 x}{3} \right]_{x=\pi/6}^{x=\pi/4} &= \sin\left(\frac{\pi}{4}\right) - \frac{1}{3} \sin^3\left(\frac{\pi}{4}\right) - \left[\sin\left(\frac{\pi}{6}\right) - \frac{1}{3} \sin^3\left(\frac{\pi}{6}\right) \right] \\ &= \frac{\sqrt{2}}{2} - \frac{1}{3} \left(\frac{\sqrt{2}}{2}\right)^3 - \left[\frac{1}{2} - \frac{1}{3} \left(\frac{1}{2}\right)^3 \right] = \frac{5\sqrt{2}}{12} - \frac{11}{24} \end{aligned}$$

Method 2: **convert limits** of integration at the time of “u” substitution

Let $u = \sin(x)$, then $du = \cos(x) dx$

Conversion of limits of integration:

$$x = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2} \quad \text{and} \quad x = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$$

$$\begin{aligned} \int_{x=\pi/6}^{x=\pi/4} [1 - \sin^2(x)][\cos(x) dx] &= \int_{u=1/2}^{u=\sqrt{2}/2} (1 - u^2) du \\ &= \left[u - \frac{u^3}{3} \right]_{1/2}^{\sqrt{2}/2} = \frac{\sqrt{2}}{2} - \frac{1}{3} \left(\frac{\sqrt{2}}{2}\right)^3 - \left[\frac{1}{2} - \frac{1}{3} \left(\frac{1}{2}\right)^3 \right] = \frac{5\sqrt{2}}{12} - \frac{11}{24} \end{aligned}$$

Example: Evaluate $\int_1^e \frac{9 \ln x}{x} dx$ let $u = \ln x$, then $du = \frac{dx}{x}$

$$\int \frac{9 \ln x}{x} dx = 9 \int \ln|x| \frac{dx}{x} = 9 \int u du = 9 \left[\frac{u^2}{2} \right] = 9 \left[\frac{(\ln x)^2}{2} \right]_1^e = 9 \left(\frac{1}{2} - 0 \right) = \frac{9}{2}$$

Example: Evaluate $\int_{-2}^2 \left(\frac{e^x + 1}{e^x} \right) dx = \int_{-2}^2 (1 + e^{-x}) dx$ let $u = -x$, then $du = -x dx$

$$\begin{aligned} \int_{-2}^2 (1 + e^{-x}) dx &= -\int (1 + e^u) du = -(u + e^u) = [x - e^{-x}]_{-2}^2 \\ &= 2 - e^{-2} - (-2 - e^2) = 4 - e^{-2} + e^2 \end{aligned}$$



Evaluate $\int_0^{\pi/2} e^{\sin x} \cos x dx$

Integrals of symmetric functions

Suppose f is continuous on $[-a, a]$

- If f is even [$f(-x) = f(x)$], then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

$$\int_{-3}^3 (-x^4 + 3) dx = 2 \int_0^3 (-x^4 + 3) dx = 2 \left[-\frac{x^5}{5} + 3x \right]_0^3 = 2 \left(-\frac{243}{5} + 9 \right) = -\frac{456}{5}$$

- If f is odd [$f(-x) = -f(x)$], then $\int_{-a}^a f(x) dx = 0$: Example: $\int_{-\pi}^{\pi} \sin(x) dx = 0$



Evaluate

$$\int_{-\pi/6}^{\pi/6} \tan^3 \theta d\theta$$

$$\int_{-4}^4 |x| dx$$



Solutions: Remember that you can check by differentiating your result!

- $\int (8 - 4x)^5 dx$ Let $u = 8 - 4x$; then $dx = -\frac{1}{4} du$
$$-\frac{1}{4} \int (8 - 4x)^5 (-4 dx) = -\frac{1}{4} \int u^5 du = -\frac{1}{4} \left(\frac{u^6}{6} \right) + C = -\frac{1}{24} u^6 + C = -\frac{1}{24} (8 - 4x)^6 + C$$
- $\int -12 \csc(3x) \cot(3x) dx$ Let $u = 3x$; then $dx = \frac{1}{3} du$
$$-4 \int \csc(3x) \cot(3x) (3 dx) = -4 \int \csc(u) \cot(u) (3 du) = 4 \csc(u) + C = 4 \csc(3x) + C$$
- $\int x^3 \sqrt{7x^4 - 5} dx$ Let $u = 7x^4 - 5$; then $du = 28x^3 dx$
$$\frac{1}{28} \int \sqrt{7x^4 - 5} (28x^3 dx) = \frac{1}{28} \int u^{1/2} du = \frac{1}{28} \left(\frac{2}{3} u^{3/2} \right) + C = \frac{1}{42} (7x^4 - 5)^{3/2} + C$$
- $\int e^{x^2+4x} (x+2) dx$ Let $u = x^2 + 4x$; then $du = (2x+4) dx$
$$\frac{1}{2} \int e^{x^2+4x} 2(x+2) dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2+4x} + C$$
- $\int 4x(2x+9)^8 dx$ Let $u = 2x+9$; then $du = 2 dx$ and $x = \frac{u-9}{2}$
$$2 \int x(2x+9)^8 (2) dx = 2 \int \frac{u-9}{2} (u)^8 du = 2 \int \left(\frac{u^9}{2} - \frac{9u^8}{2} \right) du = 2 \left(\frac{1}{2} \left(\frac{u^{10}}{10} \right) - \frac{9}{2} \left(\frac{u^9}{9} \right) \right) + C$$
$$= \frac{u^{10}}{10} - u^9 + C = \frac{(2x+9)^{10}}{10} - (2x+9)^9 + C$$
- Check: $\frac{d}{dx} \left(\frac{(2x+9)^{10}}{10} - (2x+9)^9 + C \right) = 10 \left(\frac{(2x+9)^9}{10} \right) (2) - 9(2x+9)^8 (2)$
$$= (2)(2x+9)^9 - 18(2x+9)^8 = 2(2x+9)^8 (2x+9 - 9) = 4x(2x+9)^8 \text{ VOILA!}$$
- $\int_0^{\pi/2} e^{\sin x} \cos x dx$ let $u = \sin x$; then $du = \cos x dx$
$$\int e^{\sin x} \cos x dx = \int e^u du = e^u = e^{\sin x}$$
$$\left[e^{\sin x} \right]_0^{\pi/2} = e^{\sin(\pi/2)} - e^0 = e - 1$$

- $\int_{-\pi/6}^{\pi/6} \tan^3 \theta d\theta$ Tangent is an odd function since $\tan(-\theta) = -\tan \theta$. An odd power of tangent will still be odd, and the given function is continuous on the given interval. Therefore, the answer is $\int_{-\pi/6}^{\pi/6} \tan^3 \theta d\theta = 0$.

- $\int_{-4}^4 |x| dx$ Absolute value is an even function since $|-x| = |x|$, and the function is continuous on the given interval. Therefore, $\int_{-4}^4 |x| dx = 2\int_0^4 |x| dx$.

On $[0, 4]$, $|x| = x$. So, $2\int_0^4 |x| dx = 2\int_0^4 x dx = \left[2\frac{x^2}{2} \right]_0^4 = 16$