

## INTEGRATION BY PARTS

**Objective: Evaluate integrals using integration by parts**

Formula for integration by parts:  $\int u dv = uv - \int v du$

You must be able to differentiate “u” and integrate “dv”

- Always let  $\ln(x) = u$
- Usually let a power term = u

Example: evaluate  $\int x \sin(x) dx$

- let  $u = x$  and  $dv = \sin(x)dx$   
then  $du = dx$  and  $v = -\cos(x)$
- $\int x[\sin(x)dx] = x[-\cos(x)] - \int [-\cos(x)](dx) = -x \cos(x) + \int [\cos(x)](dx)$   
 $= -x \cos(x) + \sin(x) + C$

Example: Evaluate  $\int \ln(x)dx$

- let  $u = \ln x$  and  $dv = dx$   
then  $du = \frac{dx}{x}$  and  $v = x$
- $\int \ln(x)dx = x [\ln(x)] - \int x \left( \frac{dx}{x} \right) = x \ln(x) - \int dx = x \ln(x) - x + C$

Example: Evaluate  $\int x^2 e^x dx$

- let  $u = x^2$  and  $dv = e^x dx$   
then  $du = 2x dx$  and  $v = e^x$
- $\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$
- Apply the procedure again to the remaining integral:  $\int x e^x dx$
- let  $u = x$  and  $dv = e^x dx$   
then  $du = dx$  and  $v = e^x$
- $\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$
- $\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2(x e^x - e^x) + C$   
 $= x^2 e^x - 2x e^x + 2e^x + C$

Example: Evaluate  $\int_0^1 \tan^{-1}(x) dx$

- let  $u = \tan^{-1}(x)$  and  $dv = dx$   
then  $du = \frac{dx}{1+x^2}$  and  $v = x$
- $\int_0^1 \tan^{-1}(x) dx = \left[ x \tan^{-1}(x) \right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx$
- Use substitution to evaluate the integral:  $\int_0^1 \frac{x}{1+x^2} dx$
- let  $t = 1+x^2$ , then  $dt = 2x dx$
- $\frac{1}{2} \int \frac{2x dx}{1+x^2} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln|t| = \frac{1}{2} \ln|1+x^2|$
- $\int_0^1 \tan^{-1}(x) dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \left[ \frac{1}{2} \ln|1+x^2| \right]_0^1$   
 $= \frac{\pi}{4} - \frac{1}{2} (\ln 2 - \ln 1) = \frac{\pi}{4} - \frac{\ln 2}{2}$



Evaluate:  $\int \sqrt{x} \ln x dx$

Evaluate:  $\int x^3 e^x dx$

Evaluate:  $\int (\ln x)^2 dx$

Evaluate:  $\int x \cos mx \, dx$



### Solutions

•  $\int \sqrt{x} \ln x \, dx$       Let  $u = \ln x$        $dv = \sqrt{x} \, dx$   
Then  $du = \frac{1}{x} \, dx$        $v = \frac{2}{3} x^{3/2}$

$$\int \sqrt{x} \ln x \, dx = \ln x \left( \frac{2}{3} x^{3/2} \right) - \int \frac{1}{x} \left( \frac{2}{3} x^{3/2} \right) dx$$

$$\int \frac{1}{x} \left( \frac{2}{3} x^{3/2} \right) dx = \frac{2}{3} \int x^{1/2} dx = \frac{2}{3} \left( \frac{2}{3} x^{3/2} \right) + C$$

$$\int \sqrt{x} \ln x \, dx = \ln x \left( \frac{2}{3} x^{3/2} \right) - \left( \frac{4}{9} x^{3/2} \right) + C$$

•  $\int x^3 e^x \, dx$       Let  $u = x^3$        $dv = e^x \, dx$

Then  $du = 3x^2 \, dx$        $v = e^x$

$$\int x^3 e^x \, dx = x^3 e^x - \int e^x (3x^2 \, dx) = x^3 e^x - 3 \int x^2 e^x \, dx$$

Let  $u = x^2$        $dv = e^x \, dx$

Then  $du = 2x \, dx$        $v = e^x$

$$3 \int x^2 e^x \, dx = 3x^2 e^x - 3 \int e^x (2x \, dx) = 3x^2 e^x - 6 \int x e^x \, dx$$

Let  $u = x$        $dv = e^x \, dx$

Then  $du = dx$        $v = e^x$

$$6 \int x e^x \, dx = 6x e^x - 6 \int e^x \, dx = 6x e^x - 6e^x + C$$

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

- $\int (\ln x)^2 dx$       Let  $u = (\ln x)^2$        $dv = dx$

Then  $du = 2 \ln x \left( \frac{1}{x} \right) dx$      $v = x$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2x \ln x \left( \frac{1}{x} \right) dx = x(\ln x)^2 - 2 \int \ln x dx$$

Let  $u = \ln x$        $dv = dx$

Then  $du = \frac{1}{x} dx$        $v = x$

$$2 \int \ln x dx = 2x \ln x - 2 \int dx = 2x \ln x - 2x + C$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2x \ln x + 2x + C$$

- $\int x \cos mx dx$       Let  $u = x$        $dv = \cos mx dx$

Then  $du = dx$        $v =$

Let  $U = mx$

Then  $dU = m dx$

$$\int \cos mx dx = \frac{1}{m} \int \cos mx (m dx) = \frac{1}{m} \int \cos u du = \frac{1}{m} \sin u = \frac{1}{m} \sin mx = v$$

$$\int x \cos mx dx = x \left( \frac{1}{m} \sin mx \right) - \int \frac{1}{m} \sin mx dx$$

Let  $U = mx$

Then  $dU = m dx$

$$\int \frac{1}{m} \sin mx dx = \frac{1}{m^2} \int \sin mx (m dx) = \frac{1}{m^2} \int \sin u du = -\frac{1}{m^2} \cos u = -\frac{1}{m^2} \cos mx$$

$$\int x \cos mx dx = \frac{x \sin mx}{m} + \frac{\cos mx}{m^2} + C$$