INTEGRATION USING TRIG SUBSTITUTION

Objective: Integrate Using Trig Substitution

Pythagorean trig identity: $\sin^2 x + \cos^2 x = 1$

From this identity we can quickly derive the other two related identities

- $\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 1 + \cot^2 x = \csc^2 x$
- $\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$

Example: Evaluate $\int \cos^3 x dx$

- Neither substitution nor integration by parts will work here because a factor of (sin x) would be introduced.
- Separate the cosine factor into a product and substitute using the Pythagorean identity
  - $\int \cos^3 x dx = \int (\cos^2 x \bullet \cos x) dx = \int (1 - \sin^2 x) \bullet \cos x) dx$

    $\int (1 - \sin^2 x) \bullet \cos x) dx = \int (1 - u^2) du$ let $u = \sin x \Rightarrow du = \cos x$

    $\int (1 - u^2) du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$
- General rule: Try to rewrite integrand involving odd powers of sine and/or cosine in a form which has a single sine factor or a single cosine factor. [see next section if the integrand has only even powers of sine and/or cosine.

Evaluate $\int \sin^3 x dx$

If integrand contains only even powers of sine and/or cosine, use the following ***half-angle identities***

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Example: Evaluate $\int \cos^4 x dx$

- $\int \cos^4 x dx = \int (\cos^2 x)^2 dx = \int \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx = \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)dx$

  $= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{8}\left(\frac{1}{4}\sin 4x\right) + C$

  $= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

Evaluate $\int \sin^4 x\, dx$

A trig substitution may sometimes be used to get rid of a root sign: see Example 3 on p. 401

- If you have a factor of $\sqrt{a^2 - x^2}$, let x = $a(\sin\theta)$
- If you have a factor of $\sqrt{a^2 + x^2}$, let x = $a(\tan\theta)$
- If you have a factor of $\sqrt{x^2 - a^2}$, let x = $a(\sec\theta)$

- Example: Evaluate $\int_0^2 \sqrt{4-x^2}dx$ if $0 \le \theta \le \frac{\pi}{2}$
  - Let x = $2(\sin\theta)$, then dx = $2(\cos\theta)d\theta$

    and $a^2 - x^2 = 2^2 - 2^2\sin^2\theta = 2^2(1-\sin^2\theta) = 2^2\cos^2\theta$
  - So, $\int \sqrt{2^2 - x^2}dx = \int \sqrt{(2\cos\theta)^2}\,2\cos\theta d\theta = \int (2\cos\theta)(2\cos\theta)d\theta = \int 4\cos^2\theta d\theta$
  - We now have $4\int \cos^2\theta d\theta$, which we can integrate using a half angle formula.
  - $4\int \cos^2\theta d\theta = 4\int \frac{1}{2}(1+\cos 2\theta)\,d\theta = 2(\theta + \frac{1}{2}\sin 2\theta)$
  - It is easier to convert the limits than to change everything back in terms of x!
    - If x = 0, then $\theta = \sin^{-1}\frac{x}{2} = \sin^{-1}0 = 0$
    - If x = 2, then $\theta = \sin^{-1}\frac{x}{2} = \sin^{-1}1 = \frac{\pi}{2}$
  - $\left[2(\theta + \frac{1}{2}\sin 2\theta)\right]_0^{\pi/2} = 2\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - 2\left(0 + \frac{1}{2}\sin 0\right) = \pi$

If you are finding an **indefinite integral**, you must convert back to the original variable! Problems

#9-11 fall into this category. The conversion is best done using a triangle.

Example: In #9 the substitution is $x = 3\sin\theta$, so $\theta = \sin^{-1}\frac{x}{3}$

3

$\sin\theta = \frac{x}{3}$ x

θ

$\sqrt{9-x^2}$

The indefinite integral is $\int \frac{\sqrt{9-x^2}}{x^2}dx = -\cot\theta - \theta + C$

In terms of x, $\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9-x^2}}{x^2} - \sin x^{-1}\frac{x}{3} + C$

For inquiring minds: Evaluate $\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}dt$

- Let t = $a(\sec\theta)$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, and a = 1

  [We restrict θ to these intervals so than $\tan\theta$ will be non-negative.]
- $t = \sec\theta \Rightarrow dt = \sec\theta\tan\theta\, d\theta$
- We must convert the limits of integration in this case.
  - $\sqrt{2} = \sec\theta = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ (domain of $\cos^{-1}\theta$ is $[0, \pi]$)
  - $2 = \sec\theta = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$
- $\int \frac{1}{t^3\sqrt{t^2-1}}dt = \int \frac{1}{(\sec\theta)^3\sqrt{(\sec\theta)^2-1}}\sec\theta\tan\theta\, d\theta = \int \frac{1}{(\sec\theta)^3\sqrt{(\tan\theta)^2}}\sec\theta\tan\theta\, d\theta$
- Since $\tan\theta \ge 0$, $\sqrt{(\tan\theta)^2} = |\tan\theta| = \tan\theta$

  $\int \frac{\sec\theta\tan\theta}{(\sec\theta)^3\tan\theta}d\theta = \int \frac{1}{(\sec\theta)^2}d\theta = \int (\cos\theta)^2 d\theta = \int \frac{1}{2}(1+\cos 2\theta)d\theta$ [double angle]

  $= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3} = \frac{1}{2}\left[\frac{\pi}{3} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{4} + \frac{1}{2}(1)\right]$

  $= \frac{1}{2}\left(\frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{2}\right) = \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}$

Solutions

- $\int \sin^3 x dx = \int (1-\cos^2 x)\sin x\, dx$ Let u = cos x; then du = –sinx dx

$$= -\int (1-u^2)\,du = -\left(u - \frac{1}{3}u^3\right) + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C \text{ or } \frac{1}{3}\cos^3 x - \cos x + C$$

- $\int \sin^4 x\, dx = \int (\sin^2 x)^2 dx = \int \left[\tfrac{1}{2}(1-\cos 2x)\right]^2 dx = \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)dx$

$$= \tfrac{1}{4}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{4}\int \tfrac{1}{2}(1+\cos 4x)dx = \tfrac{1}{4}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{8}\left(\tfrac{1}{4}\sin 4x\right) + C$$

$$= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$