INTEGRATION USING PARTIAL FRACTIONS

Objective: Evaluate integrals using partial fraction decomposition

Any rational function (a ratio of polynomials) can be integrated by expressing it as a **sum of simpler fractions**

- Case 1: Degree of numerator is greater than or equal to degree of denominator: perform the long division first

  Integrate $\int \frac{x^3 - 3x}{x+1}dx$ using partial fraction decomposition.

$$\begin{array}{r l} & x^2 - x - 2 \\ x+1 \overline{)} & x^3 + 0x^2 - 3x \\ & \underline{x^3 + 1x^2} \\ & \quad -x^2 - 3x \\ & \quad \underline{x^2 - x} \\ & \qquad - 2x \\ & \qquad \underline{- 2x - 2} \\ & \qquad\qquad 2 \end{array}$$

Since the denominator is linear, you could also use synthetic division.

$$\begin{array}{r|r r r r} -1 & 1 & 0 & -3 & 0 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 2 \end{array}$$

$$\int \frac{x^3 - 3x}{x+1}dx = \int \left(x^2 - x - 2 + \frac{2}{x+1} \right) dx = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 2\ln\left| x + 1 \right| + C$$

Integrate $\int \frac{x^3 + x^2 - 12x + 1}{x^2 + x - 12}dx$ using partial fraction decomposition

- Case 2: The denominator is a product of distinct linear factors, where no factor is repeated

  - Background from addition of fractions

    - $\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2) - 1(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$
    - How do we undo this process? [**painfully!**]

  - Integrate $\int \frac{x+5}{x^2+x-2}dx$ using partial fraction decomposition

    - $\frac{x+5}{x^2+x-2} = \frac{x+5}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

      [NOTE: Degree of each numerator is 1 less than degree of its denominator

      - $x + 5 = A(x+2) + B(x-1)$ Multiply both sides by LCD
      - $x + 5 = Ax + 2A + Bx - B$ Distribute
      - $x = Ax + Bx$ and $5 = 2A - B$ Equate coefficents of like powers

        $\begin{cases} 1 = A + B \\ 5 = 2A - B \end{cases} \Rightarrow A = 2,\ B = -1$ Solve for coefficients

  - $\therefore \int \frac{x+5}{x^2+x-2}dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx = 2\ln|x-1| - \ln|x+2| + C$

- Case 2: Integrate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$ using partial fraction decomposition

  - $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$

    - $x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$ Multiply both sides by LCD

      $x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$ Equate coefficents of like powers

    - $\begin{cases} 2A + B + 2C = 1 \\ 3A + 2B - C = 2 \\ -2A = -1 \end{cases} \Rightarrow A = \frac{1}{2},\ B = \frac{1}{5},\ C = -\frac{1}{10}$ Solve for coefficients

  - $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int \left(\frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}\right)dx$

    $= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$

Integrate $\int \frac{x^2 + 2x - 1}{x^2 - x} dx$

- Case 3: The denominator is a product of linear factors, some of which are repeated

  - Integrate $\int \left(\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \right) dx$ using partial fraction decomposition

    - Divide first: $\int \left(\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \right) dx = \int \left(x + 1 + \frac{4x}{x^3 - x^2 - x + 1} \right) dx$

      - Let Q(x) = $x^3 - x^2 - x + 1$. Then Q(1) = 0 implies that (x – 1) is a factor of Q(x) = $x^3 - x^2 - x + 1$, by the Remainder Theorem. Dividing shows that $x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1)$.

      - Partial fraction decomposition
        - $\frac{4x}{(x - 1)(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1}$
        - $4x = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2$ Multiply both sides by LCD
        - $4x = (A + C)x^2 + (B - 2C)x + (-A + B + C)$ Equate coefficients of like powers
        - $A + C = 0 \Rightarrow A = -C$ Solve for cofficients

          $B - 2C = 4 \Rightarrow B = 4 + 2C$

          $-A + B + C = 0 \Rightarrow C + 4 + 2C + C = 0 \Rightarrow C = -1$

          Then A = 1 and B = 2

- $$\int \left(\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \right) dx = \int \left(x + 1 + \frac{1}{x - 1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right) dx$$

$$= \frac{x^2}{2} + x + \ln|x - 1| - \frac{2}{x-1} - \ln|x + 1| + C \text{ or } \frac{x^2}{2} + x + \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

- Case 3: Background: $\frac{8}{4x+3} + \frac{1}{x^3} - \frac{2}{x} = \frac{-6x^2 + 4x + 3}{x^3(4x+3)}$
  - We know there must have been a fraction with denominator of x^3, but there may also have been fractions with denominators of x^2 and/or x.
  - Integrate $\int \frac{6}{x^3(4x+3)} dx$ using partial fraction decomposition
  - $$\frac{6}{x^3(4x+3)} = \frac{A}{4x+3} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$$

Complete this problem

- Case 4: The denominator contains irreducible quadratic factors, none of which is repeated [degree of numerator $\geq$ degree of denominator]

- Integrate $\int \left(\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} \right) dx$ using partial fraction decomposition
  - Divide if degree of numerator = degree of denominator

$$\int \left(\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} \right) dx = \int \left(1 + \frac{x - 1}{4x^2 - 4x + 3} \right) dx$$

- $4x^2 - 4x + 3$ is irreducible because its discriminant is negative; therefore it cannot be factored.

- Complete the square of the polynomial

$$4x^2 - 4x + 3 = 4\left(x^2 - x + \frac{1}{4}\right) + 3 - 1 = 2^2\left(x - \frac{1}{2}\right)^2 + 2 = (2x-1)^2 + 2$$

- Let u = 2x – 1; then du = 2dx, and $x = \frac{u+1}{2}$

- $$\int\left(1 + \frac{x-1}{4x^2 - 4x + 3}\right)dx = \int\left(1 + \frac{x-1}{(2x-1)^2+2}\right)dx = x + \frac{1}{2}\int\left(\frac{\frac{1}{2}(u+1) - 1}{u^2+2}\right)$$

$$= x + \frac{1}{4}\int\left(\frac{u-1}{u^2+2}\right)du = x + \frac{1}{4}\int\left(\frac{u}{u^2+2}\right)du - \frac{1}{4}\int\left(\frac{1}{u^2+2}\right)du$$

$$= x + \frac{1}{8}\ln(u^2+2) - \frac{1}{4} \bullet \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$ table: #17 with $a = \sqrt{2}$

$$= x + \frac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$$

- *Case 5:* The denominator contains a repeated irreducible quadratic factor

  - Integrate $\int\left(\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\right)dx$ using partial fraction decomposition

    - $$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$$
    - $1 - x + 2x^2 - x^3 = A(x^2+1)^2(Bx + C)x(x^2+1) + (Dx + E)x$
      Solution: A = 1, B = -1, C = -1, D = 1, and E = 0

- $$\int\left(\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\right)dx = \int\left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$$

  - $$\int\frac{dx}{x} - \int\frac{x}{x^2+1}dx - \int\frac{dx}{x^2+1} + \int\frac{x}{(x^2+1)^2}dx$$ let $u = x^2 + 1$

  $$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$

Solutions

- $\int \frac{x^3 + x^2 - 12x + 1}{x^2 + x - 12}dx = \int x + \frac{1}{x^2 + x - 12}dx = \frac{x^2}{2} + \int -\frac{1}{7}\left(\frac{1}{x+4}\right) + \frac{1}{7}\left(\frac{1}{x-3}\right)dx$ (cont)

$$x^2 + x - 12 \overline{\smash{\big)} x^3 + x^2 - 12x + 1}} \quad \text{quotient: } x \quad + \frac{1}{x^2 + x - 12}$$

$$\underline{x^3 + x^2 - 12x}$$

$$1$$

$$\frac{1}{x^2 + x - 12} = \frac{1}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$

$1 = A(x - 3) + B(x + 4)$

$1 = Ax - 3A + Bx + 4B = x(A + B) + (-3A + 4B)$

$0 = A + B \Rightarrow A = -B$

$1 = -3A + 4B = 3B + 4B = 7B \Rightarrow B = \frac{1}{7} \Rightarrow A = -\frac{1}{7}$