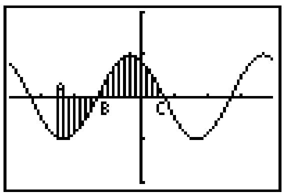
USING INTEGRATION TO FIND THE AREA BETWEEN TWO CURVES

Objective: Find the area between the graphs of two curves

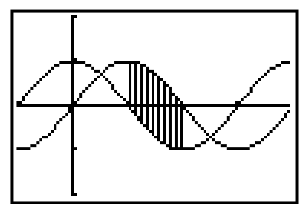
Consider the area between the graph of a **continuous curve** on [a, b] and the x-axis

- If f(x) $\geq$ 0 on [a, b], then $A = \int_a^b f(x)\,dx$
- If f(x) $\leq$ 0 on [a, b], then $A = \left|\int_a^b f(x)\,dx\right| = -\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$

Find the shaded area: 

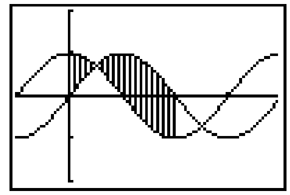
- $A = \left|\int_a^b f(x)\,dx\right| + \int_b^c f(x)\,dx$
- Analyze the problem; do NOT just put absolute values on every integral!

Find the area between the graphs of y = sin(x) and y = cos(x) on $\left[\frac{\pi}{2},\ \pi\right]$



- Always make a reasonable graph before trying to find the area between curves.
- Since the graph of y = sin(x) is above the x-axis on this interval, $A_1 = \int_{\pi/2}^{\pi} (\sin x)dx$
- Since the graph of y = cos(x) is below the x-axis on this interval, $A_2 = -\int_{\pi/2}^{\pi} \cos(x)dx$
- Therefore, A = $\int_{\pi/2}^{\pi} (\sin x)dx - \int_{\pi/2}^{\pi} \cos(x)dx \Rightarrow A = \int_{\pi/2}^{\pi} [\sin(x) - \cos(x)]\,dx$
  - NOTE: This is "top" – "bottom".
- A = $[-\cos(x) + \sin(x)]_{\pi/2}^{\pi} = -\cos(\pi) + \sin(\pi) - \left[-\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right]$ = 2
- The previous example could have been considered the area between the graphs of y = 0 and y = f(x) on [a, b] and the area between the graphs of y = f(x) and y = 0 on [b, c]..

Find the area between the graphs of y = sin(x) and y = cos(x) on $[0, \pi]$



- First find all x-values on this interval where the graphs intersect.
  - The only point where sin(x) = cos(x) on $[0, \pi]$ is at x = $\frac{\pi}{4}$
  - cos(x) $\geq$ sin(x) on $\left[0, \frac{\pi}{4}\right]$
- The area on $\left[0, \frac{\pi}{4}\right]$ is $A_1 = \int_0^{\pi/4} \cos(x)dx - \int_0^{\pi/4} \sin(x)dx$
  - sin(x) $\geq$ cos(x) on $\left[\frac{\pi}{4}, \pi\right]$
- The area on $\left[\frac{\pi}{4}, \pi\right]$ is $A_2 = \int_{\pi/4}^{\pi} \sin(x)dx - \int_{\pi/4}^{\pi} \cos(x)dx$
- The total area on $[0, \pi]$ is $\int_0^{\pi/4} [\cos(x) - \sin(x)]dx + \int_{\pi/4}^{\pi} [\sin(x) - \cos(x)]dx$

$$= \left[\sin(x) + \cos(x)\right]_0^{\pi/4} + \left[-\cos(x) - \sin(x)\right]_{\pi/4}^{\pi}$$

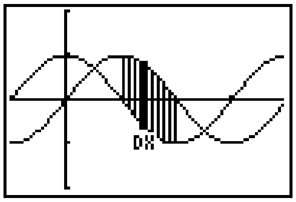
$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) + (1 - 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

If f(x) and g(x) are continuous and f(x) $\geq$ g(x) on [a, b], then the area bounded by the graphs of f(x), g(x), x = a, and x = b is $\int_a^b [f(x) - g(x)]dx$ **"top"** – **"bottom"**

Find the exact area between the graphs of y = sin(x) and $y = -\frac{2}{\pi}x + 2$ on $[0, \pi]$.

To find the area between the graphs of two continuous functions f(x) and g(x) on [a, b], where $f(x) \geq g(x)$, **draw a typical rectangle, and label its width and height**

y = cos(x)



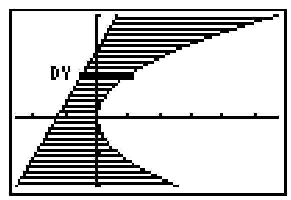
y = sin(x)

- The width of the rectangle is (dx)
- The height of the rectangle is [f(x) - g(x)].
- Its area is $\int_a^b [f(x) - g(x)]\,dx$
- This approach is going to be important for the next section!

The area between y = cos(x) and y = sin(x) on $\left[\frac{\pi}{2}, \pi\right]$ is $\int_{\pi/2}^{\pi} [\sin(x) - \cos(x)]\,dx = 2$

Find the area between the graphs of x = y^2 and $y - x = 2$ on $-2 \leq y \leq 3$

$x = y^2$



$y - x = 2$

$f(y) = y^2$ and g(y) = y - 2

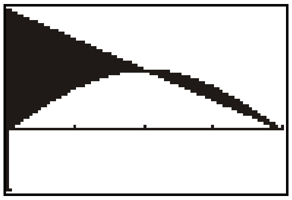
- The width of the rectangle is (dy)
- The height of the rectangle is [f(y) - g(y)].
- Its area is $\int_{y=c}^{y=d} [f(y) - g(y)]\,dy$ **"right" – "left"**

The area between $x = y^2$ and $x = y - x$ is $\int_{-2}^{3} [y^2 - (y - 2)]dy = \int_{-2}^{3} (y^2 - y + 2)dy = \frac{115}{6}$

If f(y) and g(y) are continuous and $f(y) \geq g(y)$ on [c, d], then the area bounded by the graphs of f(y), g(y), y = c, and y = d is $\int_c^d [f(y) - g(y)]\,dy$.

Find the region enclosed by the given curves: $4x + y^2 = 12,\ x = y$

Solutions

- 

[0, π] x [–1, 2]

From the graph it appears that the graphs intersect at $x = \frac{\pi}{2}$ and $x = \pi$.

Check by substitution:

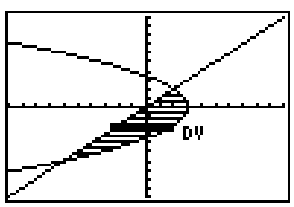
$$y = -\frac{2}{\pi}\left(\frac{\pi}{2}\right) + 2 = 1 \text{ and } \sin\left(\frac{\pi}{2}\right)$$

$$y = -\frac{2}{\pi}(\pi) + 2 = 0 \text{ and } \sin(\pi) = 0$$

$$A = \int_0^{\pi/2}\left(-\frac{2}{\pi}x + 2 - \sin x\right)dx + \int_{\pi/2}^{\pi}\left(\sin x + \frac{2}{\pi}x - 2\right)dx$$

$$= \left[-\frac{1}{\pi}x^2 + 2x + \cos x\right]_0^{\pi/2} + \left[-\cos x + \frac{1}{\pi}x^2 - 2x\right]_{\pi/2}^{\pi}$$

$$= \left(-\frac{\pi}{4} + \pi + 0\right) - (0 + 0 + 1) + (1 + \pi + 2\pi) - \left(0 + \frac{\pi}{4} - \pi\right) = \left(\frac{\pi}{2}\right)$$

- 

$x = y$

$x = \frac{12 - y^2}{4}$

Since no interval is given, we must find the points of intersection of the graphs. Since $4x + y^2 = 12$ and $x = y$, $4y + y^2 = 12$ and y = 2 and y = – 6.

$$A = \int_{y=-6}^{y=2}\left(\frac{12 - y^2}{4} - y\right)dy = \int_{y=-6}^{y=2}\left(3 - \frac{y^2}{4} - y\right)dy = \left[3y - \frac{y^3}{12} - \frac{y^2}{2}\right]_{-6}^{2}$$

$$= 6 - \frac{2}{3} - 2 - (-18 + 18 - 18) = \frac{64}{3}$$