APPLICATIONS OF INTEGRATION

6.3 Arc Length

Objective: Find the length of a curve

I. Arc length

A. Can approximate by partitioning the curve and using a Riemann sum of distances

B. $L_P = \sum_{k=1}^{n} | Q_{k-1}, \; Q_k |$

II. If f is "smooth" on [a, b], then the arc length of the graph of f from A [a, f(a)] to B [b, f(b)] is

A. $L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$ if y = f(x)

B. Find the length of the portion of the hyperbola xy = 1 from A(1, 1) to $B\left(2, \frac{1}{2}\right)$

1. $y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$

2. $L_1^2 = \int_1^2 \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\,dx \approx 1.1321$

III. If the curve is defined by x = f(y)

A. $L_c^d = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$

B. Find the length of the arc of the parabola $y^2 = x$ from C(0, 0) to D(1, 1)

1. $x = y^2 \Rightarrow \frac{dx}{dy} = 2y$

2. $L_0^1 = \int_0^1 \sqrt{1 + 4y^2}\,dy \approx 1.479$

IV. If a smooth curve with parametric equations x = f(t), y = g(t), a # t # b, is traversed exactly once as t increases from a to b, then its length is

A. $L_a^b = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

B. Find length of arc of the curve $x = t^2$, $y = t^3$ that lies between A(1, 1) and B(4, 8)

1. The parameter interval is 1 # t # 2
2. $L_1^2 = \int_1^2 \sqrt{4t^2 + 9t^4}\,dt = \int_1^2 t\sqrt{4 + 9t^2}\,dt \approx 7.6337$
3. Can evaluate using substitution or calculator