

INFINITE SEQUENCES AND SERIES

8.2 Series

Objective: Determine if a series is convergent or divergent

I. $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots, a_n, \dots\}$ is an **infinite sequence** (of terms)

II. $s_1 = a_1$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

. . .

$$s_n = a_1 + a_2 + a_3 + \dots + a_n$$

III. $\{s_n\}_{n=1}^{\infty} = a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + a_3 + \dots + a_n, \dots$ is an **infinite sequence** (of partial sums).

A. $\{s_n\}$ is a new infinite sequence, which may or may not have a limit,

B. If it has a limit, the limit is called its **sum**.

IV. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$

A. $s_n = \sum_{i=1}^n a_i = a_1 + a_2 + a_3 + a_4 + \dots + a_n$ is called its n th partial sum

B. If the sequence $\{s_n\}$ converges and $\lim_{n \rightarrow \infty} s_n = s$ exists (as a real number), then the series $\sum_{n=1}^{\infty} a_n$ is **convergent**.

The sum of the series is $a_1 + a_2 + a_3 + a_4 + \dots + a_n + \dots = s$ or $\sum_{n=1}^{\infty} a_n = s$

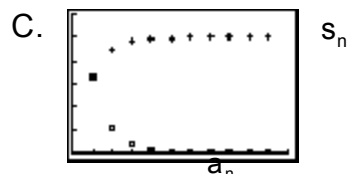
C. If the limit does not exist, then the series is **divergent**.

D. $\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$

V. Consider the series $\sum_{n=1}^{\infty} \frac{10}{3^n}$

A. $\{a_n\} = \left\{ \frac{10}{3}, \frac{10}{9}, \frac{10}{27}, \frac{10}{81}, \frac{10}{243}, \dots \right\}$

B. $\{s_n\} = \left\{ \frac{10}{3}, \frac{10}{3} + \frac{10}{9}, \frac{10}{3} + \frac{10}{9} + \frac{10}{27}, \frac{10}{3} + \frac{10}{9} + \frac{10}{27} + \frac{10}{81}, \frac{10}{3} + \frac{10}{9} + \frac{10}{27} + \frac{10}{81} + \frac{10}{243}, \dots \right\}$
 $= \left\{ \frac{10}{3}, \frac{40}{9}, \frac{130}{27}, \frac{400}{81}, \frac{1210}{243}, \dots \right\}$



NOTE: $a_1 = s_1$

$[0, 11] \times [0, 6]$

D. $\lim_{n \rightarrow \infty} a_n$ **appears** to be 0, and $\lim_{n \rightarrow \infty} s_n$ **appears** to be 5. We will revisit this problem later.

VI. Geometric series

A. Each term is obtained by multiplying the preceding term by the **common ratio** r .

$$\text{The common ratio} = \frac{a_{n+1}}{a_n}.$$

B. $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$

C. If $r = 1$, then $s_n = a + a + a + a + \dots + a = na = \pm \infty$ this geometric series diverges.

D. If $r \neq 1$, then the n th term $s_n = \frac{a(1 - r^n)}{1 - r}$.

1. $\lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}$.

2. The geometric series **converges**, and its sum is $\frac{a}{1 - r}$.

E. The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1 - r}$ if $|r| < 1$, and diverges if $|r| > 1$.

H. $\sum_{n=1}^{\infty} \frac{10}{3^n} = \sum_{n=1}^{\infty} \frac{10}{3} \left(\frac{1}{3}\right)^{n-1}$ is a geometric series with

1. $a = \frac{10}{3}$ and $r = \frac{1}{3}$

2. Since $|r| < 1$, this geometric series converges to $s = \frac{a}{1 - r} = \frac{10/3}{1 - 1/3} = 5$

VII. Write $2.\overline{317}$ as a ratio of integers

A. Using arithmetic

1. Let $x = 2.\overline{317}$

2. Then $100x = 231.\overline{717171717171} \dots$

$$\begin{array}{r} 100x = 231.\overline{717171717171} \dots \\ - x = 2.\overline{317171717171} \dots \\ \hline 99x = 229.4 \end{array}$$

$$x = \frac{229.4}{99} = \frac{1147}{495}$$

B. Using calculus

1. $2.\overline{3171717171} \dots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \dots$

2. The terms after the first term form a geometric series with $a = \frac{17}{10^3}$ and $r = \frac{1}{10^2}$

3. Since $|r| < 1$, the series converges to $s = \frac{17/10^3}{1 - 1/10^2} = \frac{17}{990}$

4. Therefore, $2.3\overline{17} = 2.3 + \frac{17}{990} = \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}$

VIII. When working with series, we consider $x^0 = 1$ even when $x = 0$! [see Example 5]

IX. See Example 6 for a use of partial fraction decomposition with a telescoping sum

X. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is called the harmonic series

A. $s_1 = 1$

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > \left(1 + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

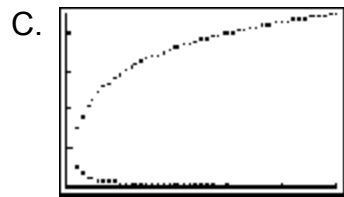
$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > \left(1 + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right)$$

$$> \left(1 + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right) = 1 + \frac{4}{2}$$

$$s_{32} > 1 + \frac{5}{2}, \quad s_{64} > 1 + \frac{6}{2}, \quad \text{and} \quad s_{2^n} > 1 + \frac{n}{2}$$

B. Therefore, $\{s_n\}$ diverges very slowly! to infinity



$$s_{50} \approx 4.499205338$$

$$s_{100} \approx 5.187377518$$

$$s_{900} \approx 7.380165881$$

$[0, 50] \times [0, 4.5]$

D. You would need to add more than a quarter of a billion terms to get a partial sum > 20 !

E. Investigate the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

XI. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$

- A. If $\sum a_n$ is convergent, the limit of the sequence $\{s_n\}$ is s (the sum of the sequence), and the limit of the sequence $\{a_n\}$ is 0
- B. The fact that $\lim_{n \rightarrow \infty} a_n = 0$ does **NOT** guarantee that $\sum a_n$ is convergent!
1. The harmonic series is an example of this
 2. $\lim_{n \rightarrow \infty} a_n = 0$ is a necessary, but not sufficient condition for the convergence of $\sum a_n$
- C. The fact that $\lim_{n \rightarrow \infty} a_n \neq 0$ does guarantee that $\sum a_n$ is divergent

XII. Test for divergence: If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent

A. Show that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$ diverges

B. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{5n^2 + 4} = \lim_{n \rightarrow \infty} \frac{1}{5 + \frac{4}{n^2}} = \frac{1}{5} \neq 0 \Rightarrow$ the series diverges.

XIII. Properties of convergent series: if $\sum a_n$ and $\sum b_n$ are convergent series, then

A. $\sum_{n=1}^{\infty} ca_n$ is convergent and $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$ [c is constant]

B. $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

C. $\sum_{n=1}^{\infty} (a_n - b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

D. See Example 9

XIV. A finite number of terms does not affect the convergence of a series

A. Suppose that it is known that $\sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$ is convergent, then

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3 + 1} \text{ is also convergent}$$

B. If it is known that the series $\sum_{n=N+1}^{\infty} a_n$ is convergent, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^N a_n + \sum_{n=N+1}^{\infty} a_n \text{ is also convergent}$$