# INFINITE SEQUENCES AND SERIES

## 8.3 The Integral and Comparison Tests; Estimating Sums

Objective: Determine whether a series is convergent or divergent

I. This section deals only with series with positive terms

A. The partial sums are increasing
B. We must determine whether the partial sums are bounded or not bounded

II. The integral test for convergence of a series $\sum_{n=1}^{\infty} a_n$

A. f(x) must be continuous, positive, and decreasing on $[1, \infty)$
B. Let $a_n = f(n)$

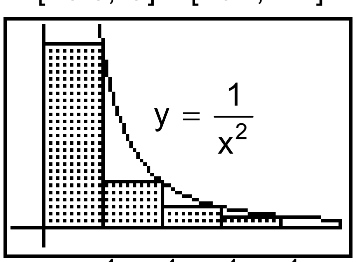
1. If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent
2. If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent

C. It is not necessary to start the series or integral at n = 1
D. f(x) must be ultimately decreasing, e.e. decreasing for x larger than some fixed number N

III. Test $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$ for convergence

A. A table of values suggests that the series converges [see p. 580]
B. Geometric argument that the series converges

[−0.5, 5] x [−0.1, 1.1]




area = $\frac{1}{1^2}$ $\frac{1}{2^2}$ $\frac{1}{3^2}$ $\frac{1}{4^2}$

1. Excluding the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = \frac{1}{x^2}$ for $x \geq 1$

a. $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{n \to \infty} \int_1^{t} \frac{1}{x^2} dx = \lim_{n \to \infty} \left[ -\frac{1}{x} \right]_1^t = 1$

b. $\sum_{i=1}^{n} \frac{1}{n^2} < \frac{1}{1^2} + \int_1^{\infty} \frac{1}{x^2} dx = 1 + 1 = 2$

2. Therefore, the partial sums are bounded, and the sum converges to a number somewhat less than 2

   Euler found the exact area to be $\frac{p^2}{6}$

C. Using the integral test

1. Let $f(x) = \frac{1}{x^2}$

   a. f is continuous and positive on $[1, \infty)$

   b. $f'(x) = -\frac{2}{x^3} < 0$ on $[1, \infty)$

2. If $\int_1^{\infty} \frac{1}{x^2} dx$ converges, then so does $\sum_{i=1}^{n} \frac{1}{n^2}$

   a. We showed above that $\int_1^{\infty} \frac{1}{x^2} dx$ converges to 1

   b. This method only tests for convergence; it does not give an estimate for the sum of a convergent series

IV. The p-series $\sum_{i=1}^{n} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$

V. The comparison test for convergence

1. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms

   a. If $\sum b_n$ is convergent and $\sum a_n \le \sum b_n$ for all n, then $\sum a_n$ is also convergent

   b. If $\sum b_n$ is divergent and $\sum a_n \ge \sum b_n$ for all n, then $\sum a_n$ is also divergent

2. The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series

3. Standard series for use with the comparison test are the p-series and the geometric series (also harmonic series)

a. p-series: $\sum \frac{1}{n^p}$ converges if p > 1 and diverges if $p \le 1$

b. geometric series: $\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \ge 1$

c. harmonic series: $\sum \frac{1}{n}$ diverges

4. The condition $a_n \le b_n$ or $a_n \ge b_n$ only needs to hold for $n \ge N$, where N is a fixed integer

   Convergence of a series is not affected by a finite number of terms!

VI. Does $\sum_{i=1}^{n} \frac{5}{2n^2 + 4n + 3}$ converge or diverge?

1. For very large n the dominant term in the denominator is $2n^2$

   a. $\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$ since $n \ge 1$

   b. $\sum_{i=1}^{n} \frac{5}{2n^2} = \frac{5}{2}\sum_{i=1}^{n} \frac{1}{n^2}$

2. Since $\sum_{i=1}^{n} \frac{1}{n^2}$ is convergent (p-series with p > 1), we conclude that

   $\sum_{i=1}^{n} \frac{5}{2n^2 + 4n + 3}$ is also convergent by the comparison test

VII. The limit comparison test

A. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms

B. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number > 0, then either both series converge or both diverge

VIII. Test the series $\sum_{i=1}^{n} \frac{1}{2^n - 1}$ for convergence

A. Let $a_n = \frac{1}{2^n - 1}$ and $b_n = \frac{1}{2^n}$

B. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$

C. $\sum_{i=1}^{n} \frac{1}{2^n} = \sum_{i=1}^{n} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1}$ is a convergent geometric series since $| r | < 1$

D. Therefore, $\sum_{i=1}^{n} \frac{1}{2^n - 1}$ is convergent by the limit comparison test

IX. Estimating the sum of a series

A. Any partial sum $s_n$ is an approximation to the sum s, but how good is the approximation?

B. The error is the size of the remainder $R_n = s - s_n$

1. The remainder is the error made when the sum of the firs n terms is used as an approximation to the total sum
2. $R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots$