

INFINITE SEQUENCES AND SERIES

8.4 Other Convergence Tests

Objective: Determine if a series is convergent or divergent

I. This section deals with series whose terms are not necessarily positive.

II. **Alternating series test:**

A. If $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$ with $b_n > 0$ satisfies

1. $b_{n+1} \leq b_n$ for all n , and

2. $\lim_{n \rightarrow \infty} b_n = 0$, then the series is convergent.

B. If the terms of an alternating series decrease to 0 in absolute value, then the series is convergent.

III. The **alternating harmonic series:** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, $b_n = \frac{1}{n} > 0$

A. $n + 1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow b_{n+1} \leq b_n$ for all n

B. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

C. The series converges by the **Alternating Series Test**.

IV. Test $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ for convergence

A. $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{3n}{4n-1} = \frac{3}{4}$, so the alternating series test does not apply.

B. Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-1)^n 3n}{4n-1}$ does not exist, the series diverges by the test for divergence.

V. The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$.

VI. **Alternating Series Estimation Theorem:** If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

- A. $0 < b_{n+1} \leq b_n$ and $\lim_{n \rightarrow \infty} b_n = 0$, then $|R_n| = |s - s_n| \leq b_{n+1}$.
- B. The error (in using s_n to approximate s) is smaller than the first neglected term.
[only applies to alternating series]

VII. Find the sum of $\sum_{i=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

A. $\frac{1}{n!} > 0$ for all n [0! = 1, by definition]

B. $\frac{1}{(n+1)!} = \frac{1}{n!(n+1)} < \frac{1}{n!}$

C. $0 < \frac{1}{n!} < \frac{1}{n}$; $\lim_{n \rightarrow \infty} \frac{1}{n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n!} = 0$

D. The series is convergent by the alternating series test.

E. $s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$

1. $b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$

2. $s_6 \approx 0.368056 \Rightarrow |s - s_6| \leq b_7 < 0.0002 \Rightarrow s \approx 0.368$ correct to three decimal places.

VIII. A series $\sum a_n$ is **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

A. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent
[p-series with $p = 2$].

B. The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent, but not absolutely convergent because $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series.
[p-series with $p = 1$]

IX. If a series $\sum a_n$ is absolutely convergent, then it is convergent. [not reversible!]

X. Test $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ for convergence

A. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} \dots$ has both positive and negative terms, but they do not alternate.

$$1. \sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

$$2. \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

3. Since $\sum \frac{1}{n^2}$ is convergent [p-series with $p = 2$], $\sum \frac{|\cos n|}{n^2}$ is convergent by the comparison test.

B. Therefore, the given series $\sum \frac{\cos n}{n^2}$ is absolutely convergent, which means it is also convergent.

XI. The ratio test

A. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

B. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

C. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the ratio test fails.

X. Test $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence

$$A. a_n = (-1)^n \frac{n^3}{3^n}$$

$$B. \left| \frac{a_{n+1}}{a_n} \right| = \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^3} = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \frac{1}{3} \left(\frac{n+1}{n} \right)^3 = \frac{1}{3} \left(1 + \frac{1}{n} \right)^3 \rightarrow \frac{1}{3} < 1$$

C. By the ratio test, the given series is absolutely convergent and therefore convergent.