

## INFINITE SEQUENCES AND SERIES

### 8.7 Taylor and Maclaurin Series

Objective: Express a given function as a power series

I. Suppose that  $f$  is any function that can be represented by a power series

$$A. f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \quad |x - a| < R$$

$$B. \text{ Its coefficients are given by } c_n = \frac{f^{(n)}(a)}{n!}. \quad [\text{See p. 608-9 for derivation}]$$

$$C. f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n. \quad [\text{Note: } 0! = 1 \text{ and } f^{(0)} = f]$$

II. Taylor series of the function  $f$  centered at  $a$ :

$$A. f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots$$

B. If  $f$  can be represented as a power series about  $a$ , then  $f$  is equal to the sum of its Taylor series.

C. There are functions that are not equal to the sum of their Taylor series.  
[see Exercise 50]

III. Maclaurin series [Special case of Taylor series with  $a = 0$ ]

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

IV. Find Maclaurin series of  $f(x) = e^x$  and its radius of convergence ["Maclaurin"  $\Rightarrow a = 0$ ]

A. If  $f(x) = e^x$ , then  $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1$  for all  $x$ .

$$B. \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

C. To find radius of convergence, let  $a_n = \frac{x^n}{n!}$

$$1. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1 \text{ for all } x.$$

2. By the Ratio Test, the series converges for all  $x$  and  $R = \infty$ .

D.  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all  $x$  and  $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$

V. Find the Taylor series for  $f(x) = e^x$  at  $a = 2$ .

A.  $f^{(n)}(2) = e^2$ ; let  $a = 2$  in definition of Taylor series.

B. Then  $\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x - 2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x - 2)^n$  for all  $x$ . [ $R = \infty$ ]

VI. If  $f(x) = T_n(x) + R_n(x)$ , where  $T_n$  is the  $n$ th-degree Taylor polynomial of  $f$  at  $a$  and  $\lim_{n \rightarrow \infty} R_n(x) = 0$  for  $|x - a| < R$ , then  $f$  is equal to the sum of its Taylor series on the interval  $|x - a| < R$ . We would need to show that  $\lim_{n \rightarrow \infty} R_n(x) = 0$ .

VII. Taylor's inequality is usually used to show  $\lim_{n \rightarrow \infty} R_n(x) = 0$ .

If  $|f^{(n+1)}(x)| \leq M$  for  $|x - a| < R$ , then the remainder  $R_n(x)$  of the Taylor series

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \text{ for } |x - a| < R.$$

VIII. Important fact:  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$  for every real number  $x$ .

IX. Important Maclaurin series

A.  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$   $I = (-1, 1)$

B.  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$   $I = (-\infty, \infty)$

C.  $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \dots$   $I = (-\infty, \infty)$

D.  $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$   $I = (-\infty, \infty)$

E.  $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \dots$   $I = [-1, 1]$

X. Use the definition of a Maclaurin series to find the Maclaurin series for  $f(x) = \frac{x}{1-x}$ .

Assume  $f$  has a power series expansion. Do not show that  $\lim_{n \rightarrow \infty} R_n(x) = 0$ .

A. Find the coefficients:

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\frac{x}{1-x}$	0
1	$\frac{1}{(1-x)^2}$	1
2	$\frac{2}{(1-x)^3}$	2
3	$\frac{3 \cdot 2}{(1-x)^4}$	$3 \cdot 2$
4	$\frac{4 \cdot 3 \cdot 2}{(1-x)^5}$	$4 \cdot 3 \cdot 2$

etc.

$$\begin{aligned}
 \text{B. } \frac{x}{1-x} &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!} + \dots \\
 &= 0 + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{24}{4!}x^4 \dots = \sum_{n=1}^{\infty} \frac{n!}{n!}x^n = \sum_{n=1}^{\infty} x^n
 \end{aligned}$$

$$\text{C. } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n \rightarrow \infty} |x| = |x| < 1 \Rightarrow R = 1$$

XI. Find the Taylor series for  $f(x) = \cos x$  at  $a = -\frac{p}{4}$

A. Find the coefficients:

$n$	$f^{(n)}(x)$	$f^{(n)}\left(-\frac{p}{4}\right)$
0	$\cos x$	$\frac{\sqrt{2}}{2}$
1	$-\sin x$	$\frac{\sqrt{2}}{2}$

2	$-\cos x$	$-\frac{\sqrt{2}}{2}$
3	$\sin x$	$-\frac{\sqrt{2}}{2}$
4	$\cos x$	$\frac{\sqrt{2}}{2}$

etc.

$$B. \cos x = f\left(-\frac{p}{4}\right) + f'\left(-\frac{p}{4}\right)\left(x + \frac{p}{4}\right) + \frac{f''\left(-\frac{p}{4}\right)}{2!}\left(x + \frac{p}{4}\right)^2 + \dots$$

$$C. \cos x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}/2}{1!}\left(x + \frac{p}{4}\right) - \frac{\sqrt{2}/2}{2!}\left(x + \frac{p}{4}\right)^2 - \frac{\sqrt{2}/2}{3!}\left(x + \frac{p}{4}\right)^3 + \frac{\sqrt{2}/2}{4!}\left(x + \frac{p}{4}\right)^4 + \dots$$

$$D. \cos x = \frac{\sqrt{2}}{2} \left[ 1 + \left(x + \frac{p}{4}\right) - \frac{1}{2!}\left(x + \frac{p}{4}\right)^2 - \frac{1}{3!}\left(x + \frac{p}{4}\right)^3 + \frac{1}{4!}\left(x + \frac{p}{4}\right)^4 + \dots \right]$$

$$E. \cos x = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{[n(n-1)]/2} \left(x + \frac{p}{4}\right)^n}{n!}$$

$$F. \text{ If } a_n = \frac{(-1)^{[n(n-1)]/2} \left(x + \frac{p}{4}\right)^n}{n!}, \text{ then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\left(x + \frac{p}{4}\right)^{n+1}}{(n+1)!} \cdot \dots \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\left(x + \frac{p}{4}\right)}{n+1} \right| = 0 < 1 \text{ for all } n \Rightarrow R = \infty$$

XII. See Example 8 for Newton's method of integration by first expressing the function in power series.

XIII. Power series can be added, subtracted, multiplied and divided like polynomials. Example 10 for multiplication and division of power series.