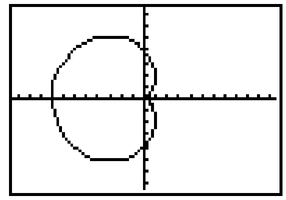
POLAR COORDINATES

G.2 Areas and Lengths in Polar Coordinates

I. Area = $\int_a^b \frac{1}{2} r^2 dq = \int_a^b \frac{1}{2}[f(q)]^2 dq$



Find area enclosed by $r = 4(1 - \cos q)$

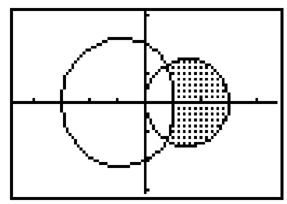
Find interval necessary to trace curve ***once***

$$A = \int_0^{2p} \frac{1}{2}[4(1 - \cos q)]dq$$

$$= 8\int_0^{2p} (1 - 2\cos q + \cos^2 q)dq = 8\int_0^{2p} [3 - 4\cos q + \cos(2q)]dq$$

$$= 8\left[3q - 4\sin q + \frac{1}{2}\sin(2q)\right]_0^{2p} = 24p$$

II. Area between two curves



Find area between $r = 3\cos q$ and $r = 2 - \cos q$

Although the first curve is traced on $[0, p]$, the second curve must be drawn on $[0, 2p]$.

Find angles at points of intersection. $3\cos q = 2 - \cos q \Rightarrow q = \frac{p}{3}, -\frac{p}{3}$

Area equals area inside first curve on minus area inside second curve on $\left[-\frac{p}{3}, \frac{p}{3}\right]$.

$$A = \int_{-p/3}^{p/2} \frac{1}{2}[(3\cos q) - (2 - \cos)^2]dq = \frac{1}{2}\int_{-p/3}^{p/2} (8\cos^2 q + 4\cos q - 4)dq$$

$$= \frac{1}{2}\int_{-p/3}^{p/2} [4\cos(2q) + 4\cos q]dq = \left[2\sin(2q) + 4\sin q\right]_{-p/3}^{p/3} = 3\sqrt{3}$$

III. Arclength: $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{dq}\right)^2}\, dq$

Find exact length of $r = 2^q$ on $[0, 2p]$

$$\int_a^b \sqrt{r^2 + \left(\frac{dr}{dq}\right)^2}\, dq = \int_0^{2p} \sqrt{(2q)^2 + [(\ln 2)2^q]^2}\, dq = \int_0^{2p} 2^q \sqrt{1 + (\ln 2)^2}\, dq$$

$$= \left[\sqrt{1 + (\ln 2)^2}\left(\frac{2^q}{\ln 2}\right)\right]_0^{2p} = \frac{\sqrt{1 + (\ln 2)^2}(2^{2p} - 1)}{\ln 2}$$