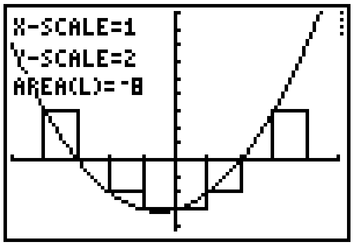
FROM AREA TO INTEGRAL: PUTTING IT ALL TOGETHER

Let $f(x) = x^2 + x - 6$

Use left endpoint approximation with n = 8 to approximate $\int_{-4}^{4}(x^2 + x - 6)dx$



- Find zeros of the function: $x^2 + x - 6 = 0 \Rightarrow (x-2)(x+3) = 0 \Rightarrow x = 2, -3$
- $\Delta x = \frac{4-(-4)}{8} = 1$
- $\int_{-4}^{-3}(x^2 + x - 6)dx \approx f(-4)(1) = 6$

  $\int_{-3}^{2}(x^2 + x - 6)dx \approx f(-3)(1) + f(-2)(1) + f(-1)(1) + f(0)(1) + f(1)(1) = -20$

  $\int_{2}^{4}(x^2 + x - 6)dx \approx f(2)(1) + f(3)(1) = 6$

  $\int_{-4}^{4}(x^2 + x - 6)dx = \int_{-4}^{-3}(x^2 + x - 6)dx + \int_{-3}^{2}(x^2 + x - 6)dx + \int_{2}^{4}(x^2 + x - 6)dx$

  Therefore, $\int_{-4}^{4}(x^2 + x - 6)dx \approx -8$
- The negative integral indicates that the area below the x-axis is larger than the area above.

NOTE: You should be able to repeat this process for right endpoint approximation (ans: 0) and midpoint approximation (ans: – 6)

Use the limit of the Riemann sum as n becomes large without bound to find the exact value of $\int_{-4}^{4}(x^2 + x - 6)dx$.

- $\Delta x = \frac{b-a}{n} = \frac{8}{n}$
- Using left endpoints: $x_0 = -4$, $x_1 = -4 + \frac{8}{n}$, $x_2 = -4 + \frac{16}{n}$, $x_3 = -4 + \frac{24}{n}$, and,

  in general, $x_i = -4 + \frac{8i}{n}$.

- $$\int_{-4}^{4}(x^2 + x - 6)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-4+\frac{8i}{n}\right)\frac{8}{n}$$

$$= \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n}\left[\left(-4+\frac{8i}{n}\right)^2 + \left(-4+\frac{8i}{n}\right) - 12\right]$$

$$= \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n}\left[16 - \frac{64i}{n} + \frac{64i^2}{n^2} - 4 + \frac{8i}{n} - 6\right]$$

$$= \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n}\left[6 - \frac{56i}{n} + \frac{64i^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\frac{8}{n}\left[\sum_{i=1}^{n} 6 - \sum_{i=1}^{n}\frac{56i}{n} + \sum_{i=1}^{n}\frac{64i^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\frac{8}{n}\left[6n - \frac{56}{n}\left(\frac{n(n+1)}{2}\right) + \frac{64}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty}\frac{8}{n}\left[6n - 28(n+1) + \frac{32(n+1)(2n+1)}{3n}\right]$$

$$= \lim_{n\to\infty}\left[48 - \frac{224(n+1)}{n} + \frac{256(n+1)(2n+1)}{3n^2}\right]$$

$$= \lim_{n\to\infty}\left[48 - 224 + \frac{224}{n} + \left(\frac{256}{3} + \frac{256}{3n}\right)\left(2 + \frac{1}{n}\right)\right]$$

$$= 48 - 224 + \left(\frac{256}{3}\right)(2) = -5\frac{1}{3}$$

NOTE: You should be able to repeat this process using right endpoints to get the ***same*** answer!

Use integration and the Evaluation Theorem to evaluate $\int_{-4}^{4}(x^2 + x - 6)dx$.

- Indefinite integral: $\int(x^2 + x - 6)dx = \frac{x^3}{3} + \frac{x^2}{2} - 6x + C$

- Definite integral: $\int_{-4}^{4}(x^2 + x - 6)dx = \left[\frac{x^3}{3} + \frac{x^2}{2} - 6x\right]_{-4}^{4}$

$$= \left[\frac{64}{3} + \frac{16}{2} - 24\right] - \left[\frac{-64}{3} + \frac{16}{2} + 24\right] = -5\frac{1}{3}$$

Aren't you glad someone discovered this nifty shortcut!