

Review for Exam 4

No credit will be given unless all steps are shown and neatly organized!

1. A **sequence** is an infinite list of numbers written in a definite order. If the n th term of a sequence has a real number limit, then the sequence converges to that number. Otherwise it converges.

Given the sequence $a_n = \frac{n}{1+n^2}$.

- Write the first 5 (exact) terms of the sequence.
- Determine if the sequence is convergent or divergent. Explain why.
- If the sequence is convergent, find its limit.

2. Do not use ***l'Hospital's Rule*** with a sequence; you may be able to use it with an associated function. If a sequence converges absolutely, then it also converges.

Given the sequence $a_n = \frac{(-1)^{n-1}2n}{3-5n}$:

- Write the first 5 (exact) terms of the sequence.
- Determine if the sequence is convergent or divergent. Explain why.
- If the sequence is convergent, find its limit.

3. A sequence is increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$ and decreasing if $a_n \geq a_{n+1}$ for all $n \geq 1$.

Given the sequence $a_n = \frac{2n-3}{3n+4}$:

- Write the first 5 (exact) terms of the sequence.
- Use the ***associated function*** to determine if the sequence is increasing or decreasing.
- Use the a_n and a_{n+1} terms to determine if the sequence is increasing or decreasing.

4. The ***Squeeze Theorem*** will not be tested on this exam.

5. A series is an infinite sequence of partial sums. If it has a limit, the limit is called its sum.

$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$ is a ***geometric series***. A ***geometric series*** is convergent if

$|r| < 1$ and divergent otherwise. The limit of a convergent geometric series is its ***sum*** $S = \frac{a}{1-r}$.

Given the series $s_n = -\frac{81}{100} + \frac{9}{10} - 1 + \frac{10}{9} - \dots$:

- Write a summation formula for the series.
- Determine if it is convergent; explain why.
- Find its sum, if possible.

6. Given the series $\sum_{n=1}^{\infty} \left(-\frac{3}{\pi}\right)^{n-1}$:

- Write the first 5 (exact) terms of the series.
- Determine if it is convergent; explain why.
- Find its sum, if possible.

7. The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$. *The harmonic series*

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges because it is a p-series with $p = 1$. The harmonic series is frequently used in comparison tests.

8. **Test for Divergence** (of a series): If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series

$\sum_{n=1}^{\infty} a_n$ is divergent. The fact that $\lim_{n \rightarrow \infty} a_n = 0$ does **NOT** guarantee that $\sum a_n$ is convergent!

9. The **Integral Test** will not be on this exam.

10. The **Comparison Test** and **Limit Comparison Test** only apply to series with all **positive** terms, however if a series is **absolutely convergent**, it is also convergent.

a) The **Comparison Test**: If $\sum b_n$ converges and $a_n \leq b_n$ for all n , then $\sum a_n$ also converges; if $\sum b_n$ diverges and $a_n \geq b_n$ for all n , then $\sum a_n$ also diverges.

b) The **Limit Comparison Test**: If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$, where c is a finite number > 0 , then both series converge or both diverge.

c) Determine if this series converges or diverges: $\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$. Tell which test you are using and why it justifies your conclusion.

d) Determine if this series converges or diverges: $\sum_{n=1}^{\infty} \frac{3^n}{5^n + n}$. Tell which test you are using and why it justifies your conclusion.

11. The **Alternating Series Test** applies only if the terms are decreasing in absolute value and the limit of the n th term is 0. The error in using the sum of the first n terms of a convergent alternating series to estimate the sum of the series is smaller than the $(n + 1)$ term, i.e. the first neglected term.

Given the series $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$:

- Determine if it converges or diverges. Tell which test you are using and why it justifies your conclusion.
- If it is convergent, estimate its sum using the first five terms.
- If it is convergent, what would be the maximum error in using the first 10 terms to estimate its sum?

Given the series $s_n = \frac{3}{6} - \frac{3}{8} + \frac{3}{10} - \frac{3}{12} + \frac{3}{14} - \dots$

- Write a summation formula for the series.
- Determine if it is convergent or divergent; explain why.
- If it is convergent, estimate its sum using the first five terms.
- If it is convergent, what is the maximum error in using the first 10 terms to estimate its sum?

12. The **Ratio Test** is the most versatile and important test in the chapter. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$, then

$\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent) if $L < 1$, but divergent if $L > 1$. The

Ratio Test fails if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$. The **Ratio Test** is used to find the open interval of convergence for power series.

a) Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^{n+1}}{(n+1)^2 4^{n+2}}$ is convergent or divergent. Explain your answer.

b) Determine if the series $\sum_{n=0}^{\infty} \frac{10^n}{n!}$ is convergent or divergent. Explain your answer.

13. $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$ is a **Power Series** centered at a .

It always converges if $x = a$. It may converge for some values of x and diverge for others. We use the **Ratio Test** to find the open interval of convergence. To find the closed interval of convergence, you must test the endpoints separately. On this test you will only be asked to find the open interval of convergence and the radius of convergence.

a) If $c_n = 1$ for all n , the power series is the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. [$R = 1$, $I = (-1, 1)$]

b) You can substitute an expression in place of “ x ” in a known power series to form a different series.

c) You can differentiate and integrate power series term-by-term to form different series.

d) Find the open interval of convergence and radius of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

e) Find the open interval of convergence and radius of convergence for $\sum_{n=0}^{\infty} n^3(x-5)^n$

f) If $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$, find a power series representation for $\frac{2}{1+9x^2}$.

Write the series in summation notation and find its open interval of convergence.

g) If $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$, use differentiation to find a power series representation for $\cos x$.

14. Find the Taylor series for $f(x) = \ln x$ centered at $a = 2$. Find the open interval of convergence and radius of convergence. Begin with the table of derivatives.

15. Find the Taylor series for $f(x) = \cos x$ centered at $a = \frac{\pi}{4}$. Find the open interval of convergence and radius of convergence. Begin with the table of derivatives.

SUMMARY OF STRATEGY FOR TESTING SERIES

The main strategy is to classify the series according to its *form*.

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent otherwise.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges otherwise. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is rational, then the series should be compared with a p-series. The value of p should be chosen by keeping only the highest powers of n in the numerator and denominator. The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n \rightarrow \infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the n th power) are often conveniently tested using the Ratio Test. Bear in mind that $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow 1$ as $n \rightarrow \infty$ for all p-series and therefore all rational or algebraic functions of n . Thus the Ratio Test should not be used for such series.
7. If $a_n = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the Integral Test is effective (assuming all the hypotheses of this test are satisfied). This is usually the test of last resort.