STRATEGY FOR TESTING SERIES

The main strategy is to classify the series according to its ***form.***

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if p > 1 and divergent otherwise.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges otherwise. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function, then the series should be compared with a p-series. The value of p should be chosen by keeping only the highest powers of n in the numerator and denominator. The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. If $a_n = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the Integral Test is effective (assuming all the hypotheses of this test are satisfied).

MIXED PRACTICE

Determine if each of the following series is convergent or divergent. If a series is a convergent geometric series, find its sum.

1. $-\frac{81}{100} + \frac{9}{10} - 1 + \frac{10}{9} - \ldots$
2. $\sum_{n=1}^{\infty} \left(-\frac{3}{\pi}\right)^{n-1}$
3. $\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n}}$
4. $\sum_{n=1}^{\infty} \frac{1}{2n+3}$
5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$
6. $\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$
7. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \ln n}$
8. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$
9. $\sum_{n=1}^{\infty} e^{-n} n!$
10. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^{n+1}}{(n+1)^2 4^{n+2}}$
11. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

12. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

13. $\sum_{n=1}^{\infty} \frac{4^n}{3^{2n-1}}$

14. $\sum_{n=1}^{\infty} \frac{n^4}{4^n}$

15. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

16. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

17. $\sum_{n=1}^{\infty} n^{-1.7}$

18. $\sum_{n=0}^{\infty} \frac{10^n}{n!}$

19. $\sum_{n=1}^{\infty} \frac{2n}{8n-5}$

20. $\sum_{n=2}^{\infty} \frac{n^3+1}{n^4-1}$

21. $\sum_{n=1}^{\infty} \frac{3^n}{5^n+n}$

22. $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n+2)}$

23. $\sum_{n=1}^{\infty} \frac{\cos\left(n/2\right)}{n^2+4n}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

ANSWERS

1. Divergent geometric series
2. Geometric series converges: $S = \frac{\pi}{\pi + 3}$
3. Divergent p-series
4. Divergent by the Integral Test
5. Divergent by the Comparison Test
6. Convergent by the Limit Comparison Test: compare with $b_n = \frac{2^n}{3^n}$
7. Convergent by the Alternating Series Test
8. Divergent by the Test for Divergence
9. Divergent by the Ratio Test
10. Divergent by the Ratio Test
11. Absolutely convergent by the Ratio Test
12. Convergent by the Comparison Test: compare with $b_n = \frac{1}{n^{3/2}}$
13. Convergent geometric series: $S = \frac{4/3}{1 - 4/9} = \frac{12}{5}$
14. Convergent by the Ratio Test
15. Convergent by Alternating Series Test
16. Convergent by Integral Test
17. Convergent p-series
18. Convergent by the Ratio Test
19. Divergent by the Test for Divergence
20. Divergent by the Limit Comparison Test: compare to $b_n = \frac{1}{n}$
21. Convergent by the Comparison Test: compare to $b_n = \frac{3^n}{5^n}$
22. Convergent by the Ratio Test
23. Absolutely convergent by the Comparison Test: compare to $b_n = \frac{1}{n^2}$
24. Convergent by the Alternating Series Test: use associated function and l'Hospital's Rule