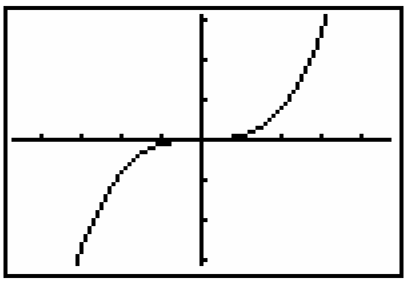
MATH 1910
Section 2.1 The Tangent and Velocity Problems

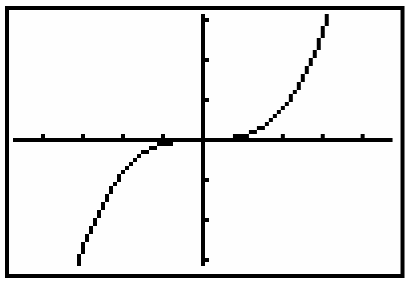
The Tangent Problem: The first problem that we'll use calculus to solve is finding the **tangent line** to a curve.

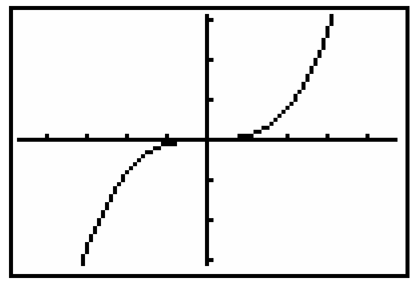
Ex. Find the equation of the tangent line to the curve $f(x) = x^3$ at the point (1 , 1).
The problem with finding the slope of a line tangent to a function's graph is that you only have one point. To find a slope of a line you need two points to use the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

What we have to do is find the various slopes of **secant lines** drawn between points on the graph. For example, if we fix one of the two points to be the (1 , 1) on the curve and use any other point on the curve, we'll have a secant line.


Secant line

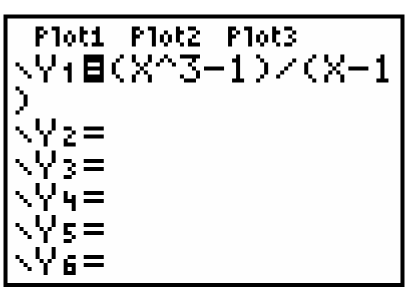

Secant line


Tangent line

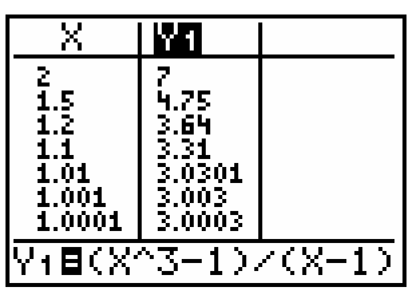
In order to find the slope of the tangent line at the point (1 , 1), we have to see what the slope of the secant lines are "approaching" as we pick points on the curve closer and closer to (1 , 1).

You can estimate secant slopes on your TI – 83 using the table. Since we're trying to find a value for $m = \frac{y_2 - y_1}{x_2 - x_1}$ when $(x_1, y_1) = (1\ ,\ 1)$ and the point (x_2, y_2) represents any other point on the graph, we know its coordinates look like this: $(x_2, y_2) = (x\ ,\ x^3)$. So the slope of any secant line through the points (1 , 1) and $(x\ ,\ x^3)$ would be $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x^3 - 1}{x - 1}$. Don't simplify this expression, leave it as it is!

Set Y1 $= \frac{x^3 - 1}{x - 1}$ and look at your table (in ASK mode) for values of the secants' slopes NEAR $x = 1$.

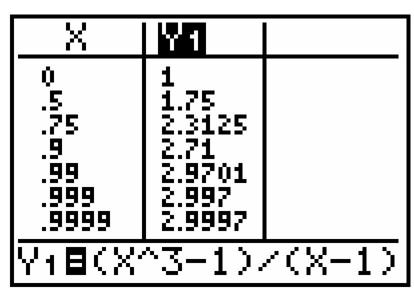


Expression for the slope of secant lines.



X	Y1
2	7
1.5	4.75
1.2	3.64
1.1	3.31
1.01	3.0301
1.001	3.003
1.0001	3.0003

Secant slopes as the point approaches from the right.



X	Y1
0	1
.5	1.75
.75	2.3125
.9	2.71
.99	2.9701
.999	2.997
.9999	2.9997

Secant slopes as the point approaches from the left.

Based on this estimate, the slope of the secant lines "approach" a value of 3. So, the slope of the tangent line could be said to be approximately $m = 3$.

MATH 1910
Section 2.1 The Tangent and Velocity Problems

The equation of the tangent line to the curve at the point (1 , 1) would be $y-1=3(x-1)$ which simplifies to $y=3x-2$ in slope – intercept form.

The Velocity Problem: now we'll take this notion of a curve's tangent and apply it to velocity problems.

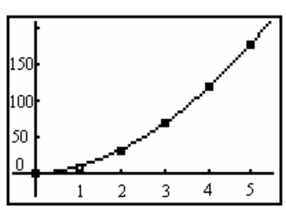
Ex. The position of a car is given by values in the table:

t (seconds)	0	1	2	3	4	5
s (feet)	0	10	32	70	119	178

a) Find the average velocity for the time period beginning when $t = 2$ and lasting for
(i) 3 sec (ii) 2 sec (iii) 1 sec
b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 2$.

SOLUTION:
We define "average velocity" as a change in distance divided by a change in time. Simply how far did someone drive the car in how many seconds? Starting at $t=2$, the car was a distance of 32 feet from its starting point. From $t=2$ until 3 seconds later when $t=5$, the car had driven a distance of 178 – 32 = 146 ft. So, this makes the average velocity of the car on that time interval to be $\frac{178-32}{5-2}=\frac{146}{3}=48.\overline{6}$ ft / sec.
From $t=2$ until 2 seconds later when $t=4$, the average velocity would be $\frac{119-32}{4-2}=\frac{87}{2}=43.5$ ft / sec.
Finally, from $t=2$ until $t=3$ the average velocity on this time interval would be $\frac{70-32}{3-2}=\frac{38}{1}=38$ ft / sec.
Our problem is, we can only use the points given to find these average velocities. One way to estimate the instantaneous velocity of the car at $t=2$ seconds is to take the average of the slopes of the secant lines before and after $t=2$. We already know the secant slope from $t=2$ to $t=3$ is 38, an average velocity of 38 ft / sec. The secant slope from $t=1$ to $t=2$ is $\frac{32-10}{2-1}=\frac{22}{1}=22$ ft / sec. So, perhaps the instantaneous velocity at $t=2$ would be exactly half way between 38 ft / sec and 22 ft / sec. These numbers average out to 30 ft / sec.
The **instantaneous velocity** of the car is defined as the "limiting value" of these average velocities taken over shorter and shorter time intervals that start at $t=2$ seconds.

Notice the similarities in this problem and the tangent problem.



We could also get a good estimate for the instantaneous velocity by plotting a curve between the points of this graph and estimating the slope of the tangent line at x = 2 using graphing paper and some good estimates for values of (x_1, y_1) and (x_2, y_2). But again, these are all estimates obtained by taking smaller and smaller increments away from the central point $t = 2$.