

MATH 1910
Limits Involving Infinity

Limits at Infinity

For any function $f(x)$ defined on some interval (a, ∞) , the limit $\lim_{x \rightarrow \infty} f(x) = L$ means that the values of $f(x)$ can be made as close to the y-value L as we like by taking x to be sufficiently large.

Ex. Find $\lim_{x \rightarrow \infty} \frac{1}{x}$ and $\lim_{x \rightarrow -\infty} \frac{1}{x}$.

SOLUTION: They both equal 0.

Here are some limits at infinity which are to your advantage to memorize:

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0 \text{ and } \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0, \text{ where } n \text{ is a positive integer}$$

$$\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2} \text{ and } \lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} e^x = 0 \text{ and } \lim_{x \rightarrow \infty} e^{-x} = 0$$

Ex. What happens to the following graphs as $x \rightarrow \infty$? (Try to do them first without using a graph)

1. $f(x) = \frac{2x-1}{x+4}$ 2. $f(x) = \frac{x+3}{3x^2-1}$ 3. $f(x) = \frac{x^2+2x-1}{x+3}$

4. $f(x) = \tan^{-1}(x)$ 5. $f(x) = \frac{2x^5-3x}{e^x}$ 6. $f(x) = e^{\frac{1}{x}}$ (tricky)

SOLUTION: 1. $f(x) \rightarrow 2$ 2. $f(x) \rightarrow 0$ 3. $f(x) \rightarrow \infty$ 4. $f(x) \rightarrow \frac{\pi}{2}$ 5. $f(x) \rightarrow 0$ 6. $f(x) \rightarrow 1$

When a graph “levels off” at either $\pm\infty$ the graph is said to reach a **horizontal asymptote**.

Ex. Evaluate $\lim_{x \rightarrow \infty} \sqrt{1+4x^2} - 2x$

SOLUTION: Use your table to see what happens when $x \rightarrow \infty$. The value should approach 0

Ex. Evaluate $\lim_{x \rightarrow \infty} \cos(3x)$

SOLUTION: Since cosine does not “level off” eventually, then there is no one limiting value as $x \rightarrow \infty$, therefore this limit does not exist.

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Infinite Limits

Previously when we saw $\lim_{x \rightarrow 0} \frac{1}{x^2}$ we concluded that the limit did not exist. But when you look at the graph of

$f(x) = \frac{1}{x^2}$ near $x = 0$ the graph shoots straight up the y-axis to infinity on both sides. Now we define this as

an infinite limit. “As $x \rightarrow 0$, the limit of $\frac{1}{x^2}$ is infinity.”

Sometimes this jump to infinity occurs on both sides, but in opposite directions as in the function $f(x) = \frac{x+1}{x-3}$.

Whenever a function goes to either ∞ or $-\infty$ at the left or right of a value $x = a$ we say that the line $x = a$ is a **vertical asymptote** for the function.

Ex. Evaluate $\lim_{x \rightarrow 3^+} \frac{x+1}{x-3}$ and $\lim_{x \rightarrow 3^-} \frac{x+1}{x-3}$

SOLUTION: Judging by the graph of this function, as $x \rightarrow 3^+$, the y-values go up the asymptote to ∞ . As $x \rightarrow 3^-$, the graph goes down the asymptote to $-\infty$. Therefore

$$\lim_{x \rightarrow 3^+} \frac{x+1}{x-3} = \infty \quad \text{and} \quad \lim_{x \rightarrow 3^-} \frac{x+1}{x-3} = -\infty$$

Two familiar functions with asymptotes are

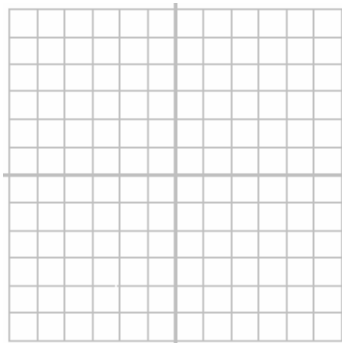
$$f(x) = \ln x,$$

vertical asymptote at $x = 0$,

Graphs of Tangent and Secant

vertical asymptotes at $x = \frac{\pi}{2} + n\pi$

Ex. Sketch a graph of a function that satisfies $\lim_{x \rightarrow -2} f(x) = \infty$, $\lim_{x \rightarrow -\infty} f(x) = 3$, $\lim_{x \rightarrow \infty} f(x) = -3$



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Ex. Evaluate $\lim_{x \rightarrow 5^+} \ln(x-5)$ (why is the $x \rightarrow 5^+$ necessary?)

Ex. Find a formula for a function that has
vertical asymptotes at $x = 1$ and $x = 3$ and a horizontal asymptote at $y = 1$.

Ex. The velocity of a falling raindrop at time t is given by $v(t) = v^*(1 - e^{-gt/v^*})$, where g is acceleration due to gravity and v^* is the terminal velocity of the raindrop.

(a) Find $\lim_{t \rightarrow \infty} v(t)$

(b) Graph $v(t)$ if $v^* = 2$ m/s and $g = 9.8$ m/s². How long does it take for the velocity of the raindrop to reach 99% of the terminal velocity?

(HINT: use XMIN = 0, XMAX = 2, YMIN = 0, YMAX = 3)