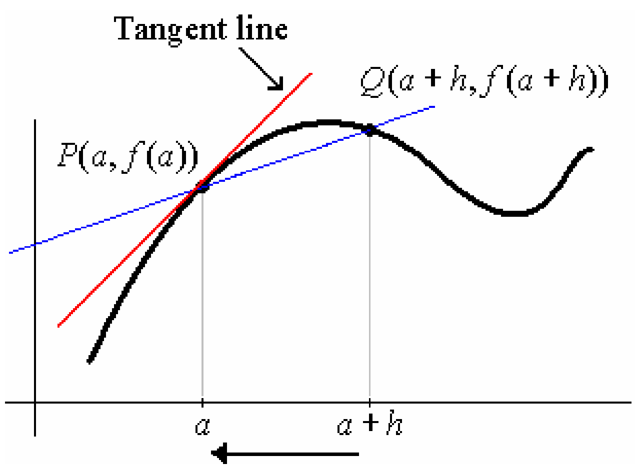
MATH 1910
Section 2.6

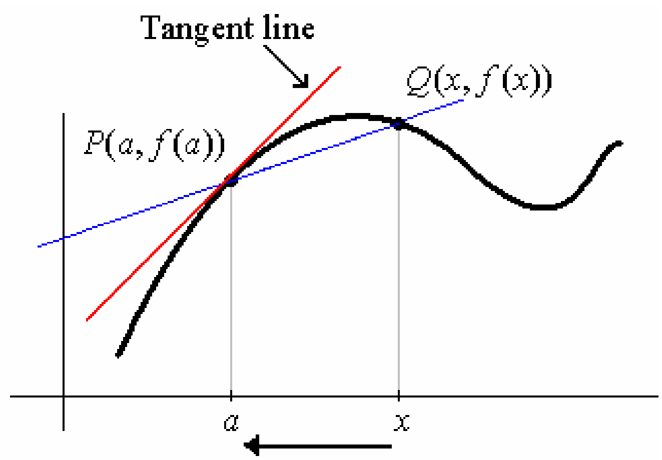
Slopes of Tangent Lines

There are two ways of defining the slope of a tangent line to a function $f(x)$ … both involve secant line slopes.
The lines formed by the points P and Q are **secant lines**, they simply connect two points on the graph of $f(x)$.
In order to find the slope of the tangent line at the point P, we have to make the point Q "move" closer to P.
Limits are used to put the point Q in motion.



Definition 1
Secant line PQ becomes the tangent line as the value of *h* approaches 0

Tangent slope = $m = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$



Definition 2
Secant line PQ becomes the tangent line as the value of *x* approaches *a*

Tangent slope = $m = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Ex) Find the equation of the tangent line to the curve $f(x) = 3x^2 - 2x + 4$ at the point $(1,\ 5)$.

SOLUTION: If you use the first definition: $m = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ … in this case $a = 1$

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \quad \rightarrow \quad = \lim_{h \to 0} \frac{\left[3(1+h)^2 - 2(1+h) + 4\right] - 5}{h}$$

$$= \lim_{h \to 0} \frac{3 + 6h + 3h^2 - 2 - 2h + 4 - 5}{h}$$

$$= \lim_{h \to 0} \frac{4h + 3h^2}{h} = \lim_{h \to 0} 4 + 3h = 4$$ ← slope at $a = 1$

The equation of the tangent line will be $y - 5 = 4(x - 1)$ → $\boxed{y = 4x + 1}$
(remember! to use the point (1, 5) on the curve with the slope m = 4)

If you use definition 2 … $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ with the value $a = 1$ …

Tangent line slope $= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$ → $= \lim_{x \to 1} \frac{(3x^2 - 2x + 4) - (5)}{x - 1}$

$$= \lim_{x \to 1} \frac{3x^2 - 2x - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(3x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1} 3x + 1 = 4$$ ← slope m = 4

Both definitions give you the slope value of m = 4

The equation of the tangent line to the function $f(x) = 3x^2 - 2x + 4$ at the point $(1,\ 5)$ is $\boxed{y = 4x + 1}$
Graph them both on the same screen to make sure this is the tangent line.

Ex) Find the equation of the tangent line to the curve $f(x) = \frac{2x - 1}{x + 7}$ at the point $(-2, -1)$.
SOLUTION: Use whichever definition you prefer … using def. 2 we get …

$m = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)}$ → $= \lim_{x \to -2} \frac{\left(\frac{2x - 1}{x + 7}\right) - (-1)}{x + 2}$ → $\frac{\left(\frac{2x - 1}{x + 7} + 1\right)}{(x + 2)} \bullet \frac{(x + 7)}{(x + 7)}$

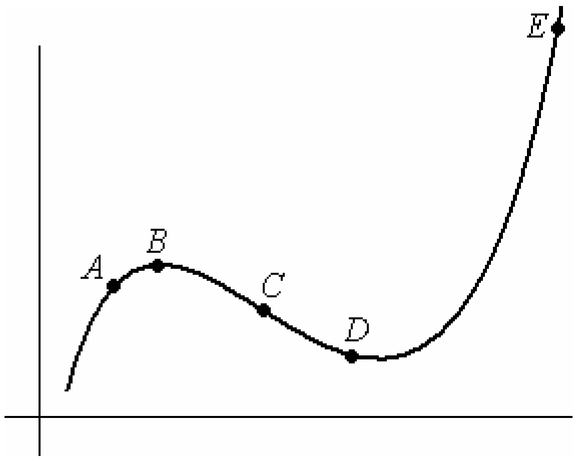
$$= \lim_{x \to -2} \frac{2x - 1 + x + 7}{(x + 2)(x + 7)}$$

$$= \lim_{x \to -2} \frac{3x + 6}{(x + 2)(x + 7)}$$

$= \lim_{x \to -2} \frac{3\cancel{(x+2)}}{\cancel{(x+2)}(x + 7)}$ → $= \lim_{x \to -2} \frac{3}{x + 7} = \frac{3}{5}$

Tangent line is … $y + 1 = \frac{3}{5}(x + 2)$ → $y = \frac{3}{5}x + \frac{1}{5}$

Ex) Consider the slope of the given curve at the five points shown. List the five slopes in increasing order and explain your reasoning.



SOLUTION: Done in class.
Try to estimate the slopes at the various points.

Velocity

Position functions can only tell us where at item is located … its location as a distance traveled or a height at time *t*.
The average velocity for an object can be found using a formula similar to the one used for secant slopes …

Average velocity between time *t* and time *a* = $\dfrac{\text{change in position}}{\text{change in time}} = \dfrac{s(t)-s(a)}{t-a}$

Instantaneous velocity is found by applying a limit to these average velocities …

Instantaneous velocity at time *a* = $\lim\limits_{h\to 0}\dfrac{s(a+h)-s(a)}{h}$ **OR** $= \lim\limits_{t\to a}\dfrac{s(t)-s(a)}{t-a}$

Ex) If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after *t* seconds is given by the function $H(t)=58t-0.83t^2$. Determine its instantaneous velocity at time $t=5$ sec.
SOLUTION: Set up the velocity limit …

Instantaneous velocity at time $t=5$ sec. = $\lim\limits_{h\to 0}\dfrac{H(5+h)-H(5)}{h}$

$$=\lim_{h\to 0}\frac{290+58h-20.75-8.3h-0.83h^2-269.25}{h}$$

$$=\lim_{h\to 0}\frac{49.7h-0.83h^2}{h}$$

$$=\lim_{h\to 0}49.7-0.83h=49.7$$ → 49.7 meters per second