MATH 1910
Section 2.7 Derivatives

In section 2.6 we defined the slope of a tangent line to a curve with equation $y = f(x)$ at the point $x = a$ to be

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

… and the velocity of an object with position function $s = f(t)$ at time $t = a$ is

$$v(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Limits of the form $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ arise in many instances when we're asked to calculate a rate of change for a particular phenomenon. Since it arises so frequently it's given its own name and notation.

Definition: The **derivative of *f* at a number *x* = *a***, denoted by $f'(a)$, is

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$$

if the limit exists.

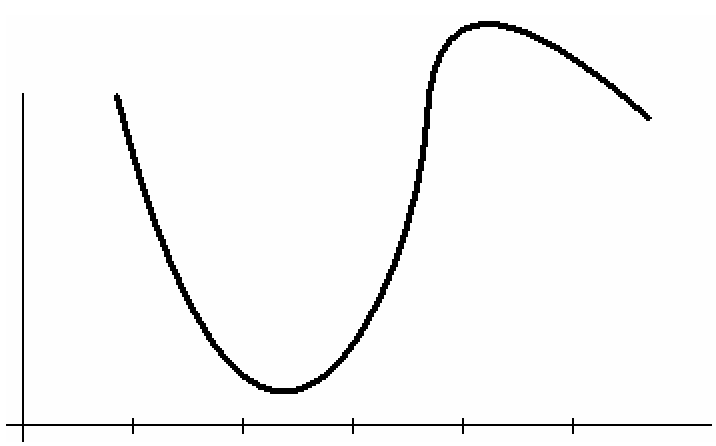
An equivalent expression for the derivative of a function at $x = a$ is $\boxed{f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}}$.

THE DERIVATIVE IS A SLOPE FUNCTION!

When you plug in an *x*-value into a function's derivative, the *y*-values you get back FROM THE DERIVATIVE tell you the slope of a tangent line to the original function at that value of *x*.

Ex. For the curve shown below, place the following quantities in increasing order.

$0 \qquad f'(1) \qquad f'(3) \qquad f'(4) \qquad f'(5)$



SOLUTION:
Your're approximating slope values.
$f'(1)$ is a negative slope and so is $f'(5)$ but since the curve is "steeper" at 1 than it is at 5, then $f'(5) > f'(1)$.
$f'(3)$ and $f'(4)$ are both on positive slopes, so analyzing the steepness suggests that $f'(3) > f'(4)$.

So, in increasing order …
$f'(1) < f'(5) < 0 < f'(4) < f'(3)$

MATH 1910
Section 2.7 Derivatives

Ex. If the tangent line to $y = f(x)$ at the point (4 , 3) passes through the point (0 , 2), find $f(4)$ and $f'(4)$.

SOLUTION: Well, the point on the curve (4 , 3) tells us that $f(4)$ = 3.
Since the tangent line passes through the points (4 , 3) and (0 , 2), the value of $f'(4)$ is simply the slope between these two points … m = 1/4. So, $f'(4)$ = 1/4

Ex. If $f(x) = 5 - 2x^2$, find $f'(a)$ for any value *a*.
Use it to find the equation of the tangent line to $y = f(x)$ at (1 , 3).

SOLUTION: Use the definition of derivative … $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \rightarrow = \lim_{h\to 0} \frac{5 - 2(a+h)^2 - 5 + 2a^2}{h}$$

$$= \lim_{h\to 0} \frac{5 - 2a^2 + 4ah + 2h^2 - 5 + 2a^2}{h}$$

$$= \lim_{h\to 0} \frac{4ah + 2h^2}{h} = \lim_{h\to 0} 4a + 2h = 4a \rightarrow \text{so } f'(a) = 4a$$

The derivative function you just found now will tell you the slope of the tangent line at any value of *x* = *a*.
The slope of the tangent line at the point (1 , 3) is found by plugging in the x value into the derivative …

THE DERIVATIVE IS A SLOPE FUNCTION!

$f'(1) = 4(1) = 4$ → slope = 4 → use the point slope formula

$y - 3 = 4(x - 1)$ → $\boxed{y = 4x - 1}$

Ex. Find $f'(a)$ for the following functions:

a) $f(x) = x^3 + 3x$ b) $f(x) = \frac{5}{x}$

SOLUTION: Try these out … a) $\boxed{f'(a) = 3a^2 + 3}$ that's the easy one …

b) Use the definition of derivative: $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \rightarrow = \lim_{h\to 0} \frac{\left(\frac{5}{a+h} - \frac{5}{a}\right)}{h}$$

$$= \lim_{h\to 0} \frac{5a - 5a - 5h}{h(a+h)(a)}$$

$$= \lim_{h\to 0} \frac{-5}{(a+h)(a)} = -\frac{5}{a^2} \rightarrow \boxed{f'(a) = -\frac{5}{a^2}}$$

MATH 1910
Section 2.7 Derivatives

Before you do this example, remember that the derivative $f'(a)$ is the **instantaneous rate of change** of $y = f(x)$ with respect to x when $x = a$. It tells you how fast the y-values are changing at that particular value of $x = a$.

Application Example

Ex. The number of bacteria (i.e. population) after t hours in a controlled laboratory experiment is $n = f(t)$.

a) What is the meaning of $f(0.5)$? What are its units?
b) What is the meaning of $f'(0.5)$? What are its units?
c) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think would be larger, $f'(5)$ or $f'(10)$?

SOLUTION:

a) $f(0.5)$ is a function value.
It represents the population of the bacteria after 0.5 hours.
Its units are just # of bacteria

b) $f'(0.5)$ is a derivative value.
It represents the rate at which the population is changing (growing) at 0.5 hours.
Its units are # of bacteria **per** hour.
(**NOTE**! Rates are always in units per unit of time, like miles PER hour, people PER year, dollars PER day)

c) Make an educated guess, but give a reason behind your guess.
In my opinion, the value of $f'(10)$ would be greater since populations usually grow at an exponential rate, so the rate of growth increases as time increases.