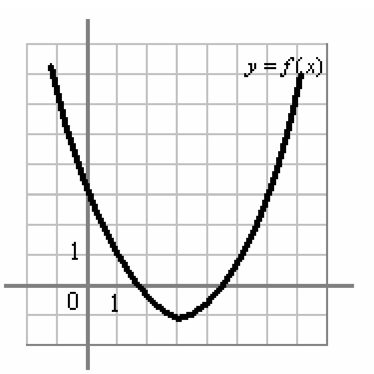
MATH 1910
Section 2.8 The Derivative as a Function

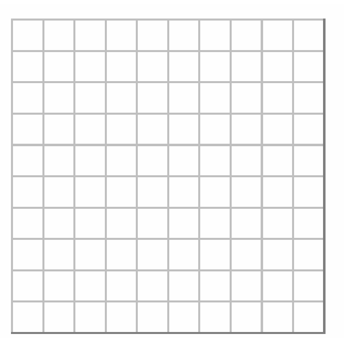
The y-values obtained from a function's derivative give the values of the "steepness" of the curve at any given point for input x. Here are some conclusions you can make about the y-values of $f'(x)$ using the graph of $f(x)$ so you could feasibly graph $f'(x)$ based on the picture you have of $f(x)$.

When the graph of $f(x)$ is …	The y-values of $f'(x)$ are …
Increasing STEEPLY	Large and positive (high above the x-axis)
Increasing SLIGHTLY	Small and positive (a little bit above the x-axis)
Decreasing STEEPLY	Large and negative (way below the x-axis)
Decreasing SLIGHTLY	Small and negative (just a little below the x-axis)
At a relative maximum, minimum or at a level point	Zero (x-intercepts)
Almost vertical or at a "V" point in the curve	Undefined (vertical asymptote)

Ex. Use the graph below to estimate the values of $f'(x)$ at the given points. Then sketch a possible graph of $f'(x)$ on the axes at the right.

$f'(0) =$ $\quad$ $f'(1) =$ $\quad$ $f'(2) =$ $\quad$ $f'(3) =$ $\quad$ $f'(4) =$ $\quad$ $f'(5) =$





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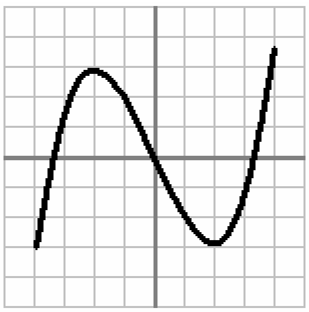
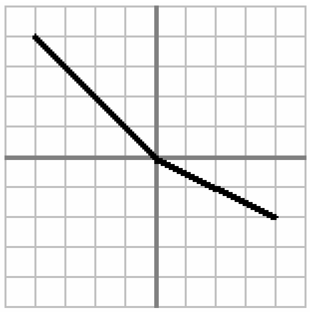
Ex. Using the following function graphs, sketch a possible graph of the function's derivative.

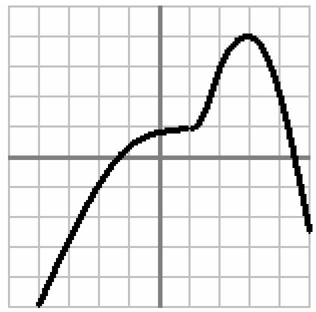
$f(x)$



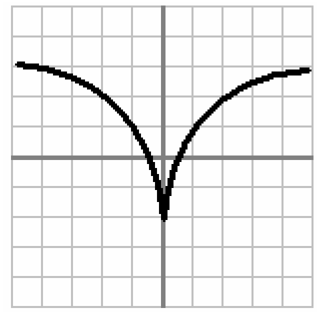
$f(x)$



$f(x)$

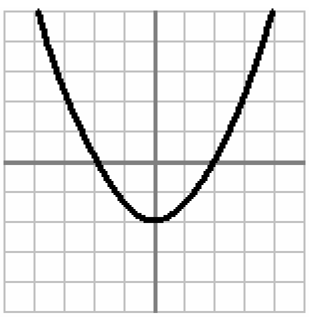


$f(x)$

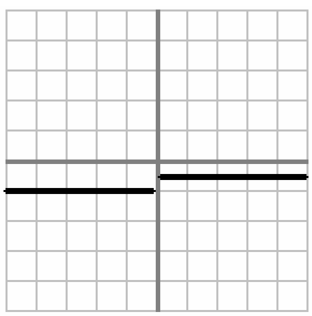


Remember, the graph of $f'(x)$ represents the various slopes the graph of $f(x)$ ranges through.

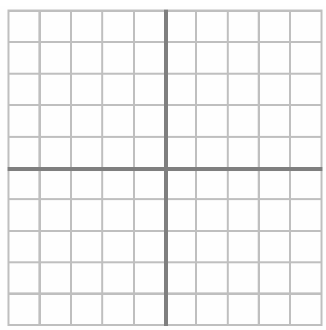
$f'(x)$



$f'(x)$

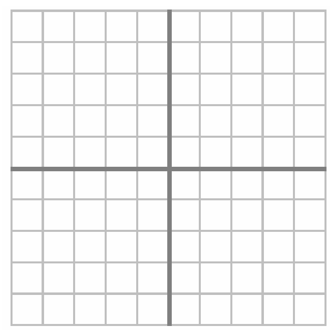


$f'(x)$



You try this one!

$f'(x)$



… and this one!

One pattern that exists for polynomial functions and their derivatives is that for a given polynomial of degree "n" its derivative is degree "$n - 1$". This should help out when you're trying to graph a polynomial function's derivative.

Since the derivative $f'(x)$ is a slope function, its values represent the range of slopes the original function $f(x)$ has on its graph.

Maximum and minimum values on the graph of $f(x)$ are points on the graph where the slope is zero. These same points produce x-intercepts on the graph of $f'(x)$.

When the graph of $f(x)$ gets very close to being vertical, the corresponding points on the graph of $f'(x)$ cause vertical asymptotes.

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Ex. Find the derivative of the given function using the definition of derivative. State the domain of the function and its derivative.

$$f(x)=\frac{x+1}{x-1}$$

SOLUTION:

Use the definition of derivative … $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \rightarrow \quad =\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}\right)-\left(\frac{x+1}{x-1}\right)}{h} \quad \leftarrow \text{simplify complex fraction}$$

$$=\lim_{h\to 0}\frac{x+1-x-h-1}{h(x+h-1)(x-1)}$$

$$=\lim_{h\to 0}\frac{-h}{h(x+h-1)(x-1)}==\lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)}=\frac{-1}{(x-1)^2}$$

So, for $f(x)=\frac{x+1}{x-1}$ the derivative is $f'(x)=\frac{-1}{(x-1)^2}$

… and they have the same domain … both domains are all real numbers except $x = 1$

Ex) Find the derivative of the function $f(x)=\sqrt{2x-5}$ using the definition of derivative.
State the domain of the function and its derivative.

SOLUTION: Apply the difference quotient limit to $f(x)=\sqrt{2x-5}$ to get $f'(x)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \rightarrow \quad =\lim_{h\to 0}\frac{\sqrt{2(x+h)-5}-\sqrt{2x-5}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{2(x+h)-5}-\sqrt{2x-5}}{h}\bullet\frac{\sqrt{2(x+h)-5}+\sqrt{2x-5}}{\sqrt{2(x+h)-5}+\sqrt{2x-5}}$$

$$=\lim_{h\to 0}\frac{2x+2h-5-2x+5}{h\left(\sqrt{2(x+h)-5}+\sqrt{2x-5}\right)}$$

$$=\lim_{h\to 0}\frac{2h}{h\left(\sqrt{2(x+h)-5}+\sqrt{2x-5}\right)}$$

$$=\lim_{h\to 0}\frac{2}{\left(\sqrt{2(x+h)-5}+\sqrt{2x-5}\right)}=\frac{1}{\sqrt{2x-5}}$$

So, for the function $f(x)=\sqrt{2x-5}$ the derivative is $f'(x)=\frac{1}{\sqrt{2x-5}}$

However, their domains are different: Domain of $f(x)$: $[5/2,\ \infty)$ Domain of $f'(x)$: $(5/2,\ \infty)$

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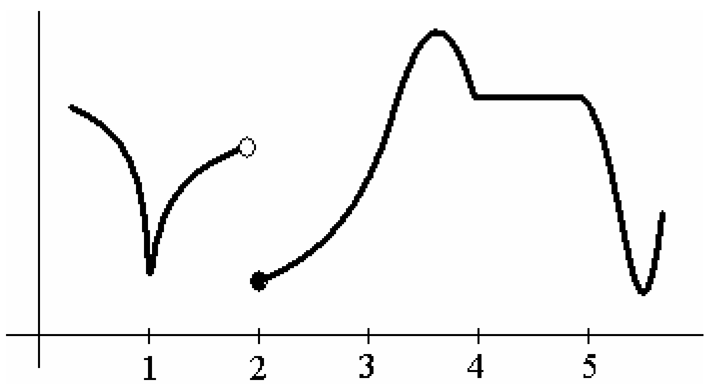
This last technicality may seem strange. When the domain of the derivative excludes values from the domain of the function, these are points where the function is said to be **non-differentiable**.

Non-differentiability occurs when …

a) the original function has a vertical tangent line (i.e. a very steep slope) at a certain value of x
b) the original function makes an abrupt change in its slope (like a kink point)
c) the original function has a vertical asymptote or a jump discontinuity in its graph.

Here's an illustration …



This function is not differentiable at …

$x = 1$ because of the "kink" in the graph

$x = 2$ because of the jump discontinuity

$x = 4$ and $x = 5$ because of the abrupt change in slope

Some common function which have these types of points are …

$f(x) = \sqrt{x}$ → because it is almost vertical at its starting point at (0, 0)

$f(x) = |\,x\,|$ → because the slope makes an abrupt change at (0, 0)

Rational functions and logarithmic functions are not differentiable at points where they have vertical asymptotes.

One handy way to judge differentiability at a point is to regard the graph of the function as a bicycle course. You want the course to be smooth and occasionally have some hills … but you don't want any sharp kinks in the path (like at x = 1 in the graph above) … or cliffs in the path (like at x = 2) … or abrupt changes in the path where your may crash your bike (like at x = 4) or any abrupt changes that may send you falling down a hill (like at x = 5).