

**MATH 1910**  
**Section 3.2 Product Rule and Quotient Rule**

**PRODUCT RULE:** If  $f(x)$  and  $g(x)$  are both differentiable functions, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$$

Using prime notation the product rule becomes

$$[f(x) \cdot g(x)]' = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Ex. Find the derivative of the function  $y = xe^x$ .

SOLUTION:

This function is the product of a single  $x$  and  $e^x$

... so according to the product rule, let  $f(x) = x$  and  $g(x) = e^x$

$$\frac{dy}{dx} = x \cdot (e^x)' + e^x \cdot (x)' = xe^x + e^x \cdot (1) = xe^x + e^x = e^x(x+1)$$

Ex. Differentiate the function  $f(x) = (x + x^2)(1 - x)$

SOLUTION:

Again, this is the product of two functions of  $x$  ... namely  $(x + x^2)$  and  $(1 - x)$

... so according to the product rule ...

$$\begin{aligned} f'(x) &= (x + x^2)(1 - x)' + (1 - x)(x + x^2)' \\ &= (x + x^2)(-1) + (1 - x)(1 + 2x) \\ &= -x - x^2 + 1 + x - 2x^2 \\ &= 1 - 3x^2 \end{aligned} \qquad f'(x) = 1 - 3x^2$$

Ex. On what interval(s) is the function  $f(x) = (x + x^2)(1 - x)$  increasing?

SOLUTION:

A function is increasing for the values of  $x$  that make the first derivative positive ... solve this inequality  $\rightarrow f'(x) > 0$

We've already found the derivative,  $f'(x) = 1 - 3x^2 \rightarrow 1 - 3x^2 > 0$

Critical points are  $x = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}}$   $\rightarrow$  Sign of  $f'(x)$

-----	-1/√3	-----	1/√3	-----
0	+++++	0	-----	-----

The signs come from the graph of  $f'(x)$

So the function  $f(x) = (x + x^2)(1 - x)$  is increasing on the interval  $(-1/\sqrt{3}, 1/\sqrt{3})$

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Ex. Product rule for 3 factors:

Show that if  $y = f(x) \cdot g(x) \cdot h(x)$  then  $y' = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$

SOLUTION:

Start by regrouping  $y = f(x) \cdot g(x) \cdot h(x)$  as  $y = [f(x) \cdot g(x)] \cdot h(x)$  and apply the product rule ...

$$\begin{aligned} y' &= [f(x) \cdot g(x)] \cdot h'(x) + [f(x) \cdot g(x)]' \cdot h(x) \\ &= [f(x) \cdot g(x)] \cdot h'(x) + [f(x) \cdot g'(x) + f'(x) \cdot g(x)] \cdot h(x) \\ &= f(x) \cdot g(x) \cdot h'(x) + f(x) \cdot g'(x) \cdot h(x) + f'(x) \cdot g(x) \cdot h(x) \quad \leftarrow \text{done!} \end{aligned}$$

Ex. What are the extrema (relative maximum/minimum values) of the function  $f(x) = x^2 e^x$ ?

SOLUTION: To find max and min points, we find the first derivative and set it equal to 0.

$$\begin{aligned} f'(x) &= x^2(e^x) + (2x)e^x = x^2 e^x + 2x e^x \quad \leftarrow \text{product rule} \\ x^2 e^x + 2x e^x &= 0 \quad \leftarrow \text{set } f'(x) = 0 \text{ to find possible maximum and minimum points} \\ x e^x (x + 2) &= 0 \quad \leftarrow \text{factor out GCF} \\ x = 0, x = -2 &\quad \leftarrow \text{since } e^x \neq 0 \end{aligned}$$

Look at the graph of  $f'(x) = x^2 e^x + 2x e^x$  relative to its x-intercepts of  $x = 0$  and  $x = -2$ .

Since  $f'(x) > 0$  (graph of  $f(x)$  is increasing) on the interval  $(-\infty, -2)$

and  $f'(x) < 0$  (graph of  $f(x)$  is decreasing) on the interval  $(-2, 0)$

and  $f'(x) > 0$  (graph of  $f(x)$  is increasing) on the interval  $(0, \infty)$  we can conclude that:

The point  $(-2, 4e^{-2})$  is a local maximum of the function and  
the point  $(0, 0)$  is a local minimum of the function.

**QUOTIENT RULE:** If  $f(x)$  and  $g(x)$  are both differentiable functions, then

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx} [f(x)] - f(x) \cdot \frac{d}{dx} [g(x)]}{[g(x)]^2}$$

Using prime notation the quotient rule becomes

$$\left[ \frac{f(x)}{g(x)} \right]' = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

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I remember the quotient rule by regarding the function as  $f(x) = \frac{TOP}{BOTTOM}$  and then the quotient rule becomes ...

$$f'(x) = (\text{BOTTOM times derivative of TOP} \text{ MINUS } \text{TOP time derivative of BOTTOM}) \text{ over } (\text{BOTTOM})^2$$

Ex. Differentiate the function  $g(x) = \frac{x}{1+x^2}$ .

SOLUTION: This function is a quotient, so its derivative is

$$g'(x) = \frac{(1+x^2)(1) - (2x)(x)}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

Ex. Simplify:  $\frac{d}{dx} \left( \frac{e^x}{1+x} \right)$

SOLUTION: Remember the symbol  $\frac{d}{dx}$  means to take the derivative of what it's applied to ...

$$\frac{d}{dx} \left( \frac{e^x}{1+x} \right) = \frac{(1+x)(e^x) - (e^x)(1)}{(1+x)^2} = \frac{e^x + xe^x - e^x}{(1+x)^2} = \frac{xe^x}{(1+x)^2}$$

Ex. Determine the interval(s) where the function  $f(x) = \frac{e^x}{x}$  is concave up by finding its second derivative and analyzing its values.

SOLUTION: Apply the quotient rule to get  $f'(x)$  first ...  $f'(x) = \frac{(x)(e^x) - (e^x)(1)}{x^2} = \frac{xe^x - e^x}{x^2}$

$$\begin{aligned} \text{Differentiate } f'(x) = \frac{xe^x - e^x}{x^2} \text{ to get the second derivative ... } f''(x) &= \frac{(x^2)(xe^x - e^x)' - (xe^x - e^x)(x^2)'}{x^4} \\ &= \frac{(x^2)(xe^x + e^x(1) - e^x) - (xe^x - e^x)(2x)}{x^4} \\ &= \frac{x^3e^x - 2x^2e^x + 2xe^x}{x^4} \\ &= \frac{e^x(x^2 - 2x + 2)}{x^3} \end{aligned}$$

The graph of  $f''(x)$  is positive on the interval  $(0, \infty)$  ... so  $f(x) = \frac{e^x}{x}$  is concave up on the interval  $(0, \infty)$ .

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A good mnemonic for the quotient rule is to call the numerator of the function “HI” and the denominator of the function “HO”, as in

$f(x) = \frac{\text{HI}}{\text{HO}}$  (HI is always on top). The derivative found by the quotient rule is “HO DE HI minus HI DE HO all over HO HO”

where the word “DE” tells you where to differentiate.

Ex. Find the equation of the line tangent to the curve  $h(x) = \frac{\sqrt{x}}{x-3}$  at the point (4, 2)

SOLUTION: Remember to rewrite the function as  $h(x) = \frac{x^{1/2}}{x-3}$

$$\begin{aligned} h'(x) &= \frac{\text{HO DE HI} - \text{HI DE HO}}{\text{HO HO}} \\ &= \frac{(x-3) \cdot (1/2)x^{-1/2} - x^{1/2} \cdot (1)}{(x-3)^2} \\ &= \frac{x^{-1/2}[0.5x - 1.5 - x]}{(x-3)^2} \\ &= \frac{[-0.5x - 1.5]}{\sqrt{x} \cdot (x-3)^2} \end{aligned}$$

Evaluate  $h'(4)$  to get the slope of the tangent line:  $h'(4) = \frac{-4-3}{2 \cdot 2 \cdot 1} = -\frac{7}{4}$

Thus the tangent line at the point (4, 2) is  $y - 2 = -\frac{7}{4}(x - 4) \rightarrow y = -\frac{7}{4}x + 9$

Ex. If  $f(3) = 4$ ,  $g(3) = 2$ ,  $f'(3) = -6$ , and  $g'(3) = 5$ , find the following numbers:

a)  $(fg)'(3)$

b)  $\left(\frac{f}{g}\right)'(3)$

SOLUTION:

a)  $(fg)' = f \cdot g' + g \cdot f'$

so  $(fg)'(3)$  means use input value of 3

$$(fg)'(3) = f(3) \cdot g'(3) + g(3) \cdot f'(3)$$

$$= 4 \cdot 5 + 2 \cdot (-6)$$

$$= 8$$

b)  $\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$

input the value 3 ...

$$\left(\frac{f}{g}\right)'(3) = \frac{g(3) \cdot f'(3) - f(3) \cdot g'(3)}{[g(3)]^2}$$

$$= \frac{2 \cdot (-6) - 4 \cdot 5}{[2]^2}$$

$$= -8$$