

MATH 1910**Section 3.3 Rates of Change in the Natural and Social Sciences**

Ex. The position of a particle moving on the x axis is modeled by the equation $s(t) = 36t - 12t^2 + t^3$, $t \geq 0$, where t is measured in seconds and $s(t)$ is measured in inches.

- What is the velocity of the particle at time t ?
- What is the velocity after 2 seconds?
- When is the particle at rest?
- When is the particle moving forward (to the right)?
- Find the total distance traveled in the first 10 seconds.
- Draw a one-dimensional position diagram showing the path of the particle as it moves on the x -axis.
- Find the acceleration of the particle at time t and after 2 seconds.
- Graph all three functions: position $s(t)$, velocity $s'(t)$ and acceleration $s''(t)$.
- When is the particle speeding up? When is it slowing down?

SOLUTION: Here's what we'll need for this problem ...

Position function $\rightarrow s(t) = 36t - 12t^2 + t^3$, $t \geq 0$ (the inequality is just to let you know to only use positive time value)

velocity function $\rightarrow v(t) = s'(t) = 36 - 24t + 3t^2$ (first derivative)

acceleration function $\rightarrow a(t) = v'(t) = s''(t) = -24 + 6t$ (second derivative)

a) Velocity at time t is $s'(t) = 36 - 24t + 3t^2$.

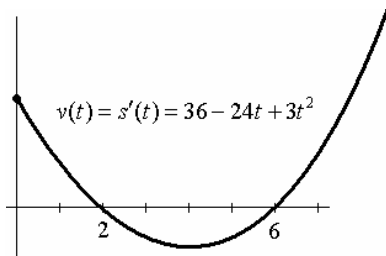
b) Velocity after 2 seconds is $v(2) = s'(2) = 36 - 24(2) + 3(2)^2 = 0$ inches / second

c) The particle is said to be "at rest" when the velocity is zero ... so set velocity = 0 and solve for time t .

$$36 - 24t + 3t^2 = 0 \quad \rightarrow \quad \text{solutions are at } t = 2 \text{ sec. and } t = 6 \text{ sec.}$$

This particle is at rest at $t = 2$ sec. and $t = 6$ sec.

d) "Right" and "up" should always be considered a positive vector velocity value. "Left" and "down" are negative vector velocity values.



This particle is moving right when $s'(t) > 0$

... look at the graph of $s'(t) = 36 - 24t + 3t^2$

The velocity is positive on these intervals: $(0, 2) \cup (6, \infty)$

This means the particle is moving right on the same intervals ...
 $(0, 2) \cup (6, \infty)$ seconds.

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e) Be careful with this one ... in the first 10 seconds of its travel, the particle turns around twice ... once at 2 seconds and again at 6 seconds. You need to find the total distance in intervals ...

$s(0) = 0$ inches ← the particle starts at 0 inches.

$s(2) = 32$ inches ← the particle is 32 inches to the right after 2 seconds

$s(6) = 0$ inches ← the particle is back at 0 inches ... where it started.

$s(10) = 160$ inches ← at 10 seconds, the particle is 160 inches to the right of its starting point.

So the total distance traveled is $32 + 32 + 160 = \mathbf{224}$ inches

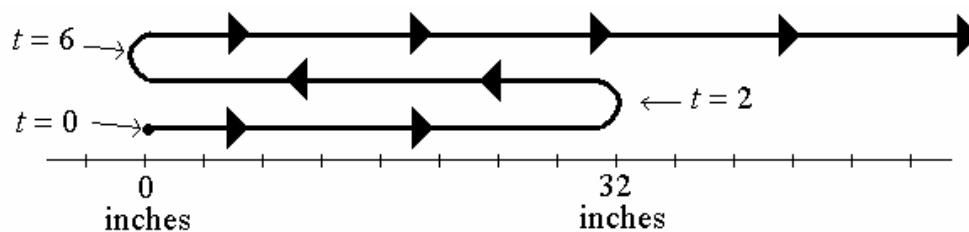
f) The derivative graph helps to show the motion of the particle ...

moving right between $t = 0$ and $t = 2$ seconds ...

moving left between $t = 2$ and $t = 6$ seconds ...

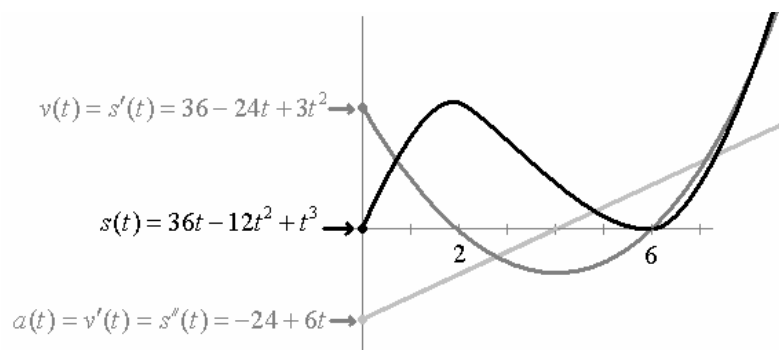
moving right after 6 seconds

using the position values from part (e) we can represent the actually 1-dimensional motion of this particle ...



g) The acceleration after 2 seconds is $a(2) = s''(2) = -24 + 6(2) = -12$ inches/sec²

h)



g) The particle is “speeding up” when its acceleration and velocity have the same sign ...

Speeding up on the time interval between $t = 2$ and $t = 4$ seconds ... and again after $t = 6$ seconds to infinity.

The particle is slowing down when the velocity and acceleration have opposite signs

Slowing down between $t = 0$ and $t = 2$ seconds ... and between $t = 4$ and $t = 6$ seconds.

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Ex. A spherical balloon is being inflated. Find the rate of change of the volume, $V = \frac{4\pi r^3}{3}$ with respect to the radius r when r is (a) 10 cm (b) 11 cm (c) 12 cm. What conclusions can you make?

SOLUTION:

The function $V = \frac{4\pi r^3}{3}$ defines the volume of the balloon as a function of its radius.

The derivative of this function gives the rate of change of volume with respect to the radius.

$$V = \frac{4\pi r^3}{3} \quad \rightarrow \quad V' = \frac{4\pi}{3}(3r^2)$$
$$V' = 4\pi r^2$$

(a) $V'(10) = 4\pi(10)^2 = 400\pi \text{ cm}^3 \text{ per cm}$

(meaning the volume is growing at a rate of $400\pi \text{ cm}^3$ per cm of radius length when the radius is 10 cm long.)

(b) $V'(11) = 4\pi(11)^2 = 484\pi \text{ cm}^3 \text{ per cm}$

(c) $V'(12) = 4\pi(12)^2 = 576\pi \text{ cm}^3 \text{ per cm}$

Since the rate of change is increasing as the radius increases ... the volume is obviously growing, but it is growing at an increasing rate.

Ex. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure (in atmospheres, atm) and the volume remains constant: $PV = C$.

(a) Find the rate of change of the pressure with respect to the volume.

(b) What does the sign of the result from part (a) signify?

SOLUTION:

(a) $PV = C \quad \rightarrow$ rewrite so that pressure is a function of volume $\rightarrow \quad P = \frac{C}{V}$

The rate of change of pressure with respect to volume is $\rightarrow P' = C(-V^{-2}) \rightarrow P' = -\frac{C}{V^2}$

(b) The negative sign indicates the fact that pressure (as a function of volume) is a decreasing function.

The more volume increases, the more pressure decreases which is also suggested by the inverse proportional relation $P = \frac{C}{V}$

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Ex. Assume a certain physical phenomenon is modeled by the equation $F = \frac{5A^2}{\sqrt{BC}}$. Find ...

(a) The rate of change of F with respect to A , $\frac{dF}{dA}$ (this means assume that B and C are constants)

(b) The rate of change of F with respect to B , $\frac{dF}{dB}$ (this means assume that A and C are constants)

SOLUTION:

(a) Isolate the variable $A \rightarrow F = \frac{5}{\sqrt{BC}} \cdot A^2 \rightarrow \frac{dF}{dA} = \frac{5}{\sqrt{BC}} \cdot (2A) = \frac{10A}{\sqrt{BC}}$

(b) Isolate the variable $B \rightarrow F = \frac{5A^2}{\sqrt{C}} \cdot B^{-1/2} \rightarrow \frac{dF}{dB} = \frac{5A^2}{\sqrt{C}} \cdot \left(\frac{1}{2} B^{-3/2}\right) = \frac{5A^2}{\sqrt{C}} \cdot \frac{1}{2\sqrt{B^3}} = \frac{5A^2}{2\sqrt{B^3C}}$