MATH 1910
Section 3.4 Derivatives of Trigonometric Functions

The Derivative of Sine

To differentiate the function $f(x) = \sin x$, we'll start with its difference quotient limit:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

The expression $\sin(x+h)$ can be expanded using a formula from trig: $\sin(x+h) = \sin(x)\cos(h)+\cos(x)\sin(h)$

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

Rearrange the terms up top and then separate the numerator:

$$\lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h}+\frac{\cos(x)\sin(h)}{h}$$

After some further algebraic rearranging, this limit becomes

$$\lim_{h\to 0}\ \sin(x)\cdot\left(\frac{\cos(h)-1}{h}\right)+\cos(x)\cdot\left(\frac{\sin(h)}{h}\right)$$

Since $\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)=0$ and $\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)=1$, the difference quotient limit becomes

$$\lim_{h\to 0}\ \sin(x)\cdot\left(\frac{\cos(h)-1}{h}\right)+\cos(x)\cdot\left(\frac{\sin(h)}{h}\right) = \sin(x)\cdot(0)+\cos(x)\cdot(1) = \cos x$$

After all the calculations, we have finally shown that the derivative of $f(x)=\sin x$ is $f'(x)=\cos x$
(The proof for the derivative of $f(x)=\cos x$ is similar.)

Derivatives of the Trigonometric Functions (KNOW THEM WELL!)

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$

Ex. Show that $\frac{d}{dx}(\tan x) = \sec^2 x$

SOLUTION:
Any trigonometric function's derivative can be found knowing only the derivatives for sine and cosine.

Since $\tan x = \dfrac{\sin x}{\cos x}$, to get the derivative of tangent, we simply need to apply the quotient rule to the expression $\dfrac{\sin x}{\cos x}$.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Ex. Differentiate $f(x) = 5x^3 \sin x$

SOLUTION: This is a product of two functions, the product rule is in order:

$$f'(x) = (5x^3)(\cos x) + (15x^2)(\sin x) = 5x^3 \cos x + 15x^2 \sin x$$

Some algebraic simplification makes $f'(x) = 5x^2(x\cos x + 3\sin x)$

Ex. Differentiate $g(x) = \dfrac{\sec x}{1 + \tan x}$.

SOLUTION: First and foremost this function is a quotient, therefore we use the quotient rule first.

$$g'(x) = \frac{\text{bottom} \cdot \text{top derivative} \; - \; \text{top} \cdot \text{bottom derivative}}{\text{bottom squared}} = \frac{(1+\tan x)\cdot \sec x \tan x - (\sec x)(\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2} = \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} \quad \text{(recall that } \sec^2 x - \tan^2 x = 1\text{)}$$

$$\boxed{= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}}$$ ← done!

Ex. For what values of *x* does $f(x) = x - 2\cos x$ have a horizontal tangent?

SOLUTION: Remember, horizontal tangents occur when $f'(x) = 0$. Find the first derivative … $f'(x) = 1 + 2\sin x$

Set the first derivative equal to 0 and find the critical points → $f'(x) = 1 + 2\sin x = 0$

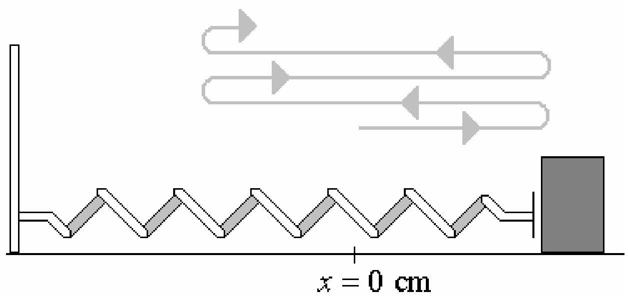
$$\sin x = -\frac{1}{2}$$

(you need to recall your unit circle here) $x = \dfrac{7\pi}{6}$ and $\dfrac{11\pi}{6}$

In general, the graph of $f(x) = x - 2\cos x$ will have a horizontal tangent at the coterminal values, too …

$$x = \frac{7\pi}{6} + 2n\pi \text{ and } \frac{11\pi}{6} + 2n\pi$$

Ex.



A mass on a spring vibrates horizontally on a smooth level surface as shown in the figure at the left.

The equation of motion for the mass is $x(t) = 8\sin t$, where x is in centimeters and t is in seconds.

a) Find the velocity and acceleration at time t.
b) Find the position, velocity and acceleration of the mass at $t = 2\pi/3$ seconds.
What is its direction of motion at that time?
Is it speeding up or slowing down?

SOLUTION:
a) Since the position is defined by the function $x(t) = 8\sin t$, the velocity of the mass on the end of the spring would be $x'(t) = 8\cos t$ and the acceleration at time t would be $x''(t) = -8\sin t$.

b) At time $t = \frac{2\pi}{3}$,

the **position** is going to be $x(\frac{2\pi}{3}) = 8\sin(\frac{2\pi}{3}) = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$ centimeters to the right of the equilibrium position

The **velocity** will be $x'(\frac{2\pi}{3}) = 8\cos(\frac{2\pi}{3}) = 8 \cdot (-\frac{1}{2}) = -4 \text{ cm/sec}$ moving to the left (why left?)

The **acceleration** of the mass will be $x''(\frac{2\pi}{3}) = -8\sin(\frac{2\pi}{3}) = -8 \cdot (\frac{\sqrt{3}}{2}) = -4\sqrt{3} \text{ cm/sec}^2$

… and since the velocity and acceleration functions have the same sign here, the mass is speeding up at this time.

Ex. What is the 51st derivative of $y = \sin x$?
SOLUTION:

$y = \sin x$
$y' = \cos x$
$y'' = -\sin x$
$y''' = -\cos x$
$y^{iv} = \sin x$ ← the derivatives are periodic with a period of 4.

This means the 48th derivative of $y = \sin x$ is also $\sin x$ …

$y^{48} = \sin x$
$y^{49} = \cos x$
$y^{50} = -\sin x$
$y^{51} = -\cos x$ → The 51st derivative of $y = \sin x$ is $y^{51} = -\cos x$