The Chain Rule:
The trick to being successful with the chain rule is being able to decompose functions by recognizing one function as an "outer" function and another as the "inner" function.

For example in the function $f(x)=\sqrt{2x-5}$ you could split this operation in two parts: the "$2x-5$" is considered the inner function and the "$\sqrt{\ }$" is considered the outer function.

Here's the way the chain rule works: If f and g are both differentiable functions and $F=f\circ g$ is the composite function defined by $F(x)=f(g(x))$, then F is differentiable and F' is given by the product

$$\boxed{F'(x)=f'(g(x))\cdot g'(x)}$$ ← **The Chain Rule**

Ex. Find $f'(x)$ if $f(x)=\sqrt{2x-5}$.

SOLUTION: Always rewrite radicals using fraction exponents: $f(x)=(2x-5)^{1/2}$
According to the chain rule,

First you differentiate the outer function leaving the inner function alone

The derivative of the outer function is $\frac{1}{2}(2x-5)^{-1/2}$

Second, you multiply by the derivative of the inner function

The derivative of the inner function $2x-5$ is 2.

The total derivative is written as $f'(x)=\frac{1}{2}(2x-5)^{-1/2}\cdot 2$ → $f'(x)=(2x-5)^{-1/2}$

Ex. Differentiate $f(x)=(3x^2+2x-4)^8$
SOLUTION: Chain rule:

First you differentiate the outer function leaving the inner function alone

The derivative of the outer function is $8(3x^2+2x-4)^7$ using the power rule

Second, you multiply by the derivative of the inner function

The derivative of the inner function is $6x+2$

The total derivative is $f'(x)=8(3x^2+2x-4)^7\cdot(6x+2)$

Because of the exponent 7 you can't do any FOIL-ing or distributing. Leave the answer just like this.

Ex. Find the derivatives of the following functions using the chain rule.

a) $y = \cos(x^2)$ b) $y = \cos^2 x$

SOLUTION:

a) It's obvious which is the inner and the outer function. The outside function is cosine and the inner function is x^2.

First you differentiate the outer function leaving the inner function alone

The derivative of the outer function is $-\sin(x^2)$

Second, you multiply by the derivative of the inner function

The derivative of the inner function is $2x$

The total derivative is $f'(x) = -\sin(x^2) \cdot 2x = \ -2x\sin(x^2)$

b) It might help to rewrite it as $y = (\cos x)^2$. Now the outer function is the squaring function and the inner function is cosine

First you differentiate the outer function leaving the inner function alone

The derivative of the outer function is $2(\cos x)^1 = 2\cos x$

Second you multiply by the derivative of the inner function

The derivative of the inner function is $-\sin x$

The total derivative is $y' = (2\cos x)\cdot(-\sin x) = \ -2\cos x \sin x$

Ex. Differentiate $y = (5x^3 - 2)^{239}$

SOLUTION

Outer derivative: $239(5x^3 - 2)^{238}$ Inner derivative: $15x^2$

Total derivative: $y' = 239(5x^3 - 2)^{238} \cdot 15x^2 = 3585x^2(5x^3 - 2)^{238}$

Ex. $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt[3]{x^2 + 2x - 1}}\right)$

SOLUTION:

First, rewrite the function as $f(x) = (x^2 + 2x - 1)^{-1/3}$

Outer derivative: $-\frac{1}{3}(x^2 + 2x - 1)^{-4/3}$ Inner derivative: $2x + 2$

Total derivative: $f'(x) = -\frac{1}{3}(x^2 + 2x - 1)^{-4/3}(2x + 2)$

Ex. Differentiate $y = e^{\tan x}$

SOLUTION:

Outer derivative: $e^{\tan x}$ Inner derivative: $\sec^2 x$

Total derivative: $y' = e^{\tan x} \cdot \sec^2 x$

Ex. Find the derivative of $f(x) = (3x-4)^5(2x^2-3x+5)^6$.

SOLUTION: This is a product, so first you apply the product rule.

$$f'(x) = (3x-4)^5 \cdot \left[\tfrac{d}{dx}(2x^2-3x+5)^6\right] + (2x^2-3x+5)^6 \cdot \left[\tfrac{d}{dx}(3x-4)^5\right] \leftarrow \text{product rule}$$

Then to differentiate the functions in brackets, you use the chain rule on them

$$= (3x-4)^5 \cdot \left[6(2x^2-3x+5)^5 \cdot (4x-3)\right] + (2x^2-3x+5)^6 \cdot \left[5(3x-4)^4 \cdot (3)\right]$$

↑ CHAIN RULE ↑ CHAIN RULE

Factor and simplify:

$$= (3x-4)^4(2x^2-3x+5)^5 \cdot \left[6(3x-4)(4x-3) + 15(2x^2-3x+5)\right]$$

$$= (3x-4)^4(2x^2-3x+5)^5 \cdot \left[72x^2-150x+72+30x^2-45x+75\right] =$$

$$= (3x-4)^4(2x^2-3x+5)^5 \cdot \left[102x^2-195x+147\right] \leftarrow \text{done}$$

Ex. Differentiate $y = \sin(e^{x^2-1})$

SOLUTION: This is a composition of 3 functions, differentiate from the outside in

The derivative of the outermost function is $\cos(e^{x^2-1})$

The derivative of the middle function, which is "*e*" is e^{x^2-1}

The derivative of the innermost function is for the function $x^2-1 \rightarrow 2x$

The total derivative is the product of these three parts

$$y' = \cos(e^{x^2-1}) \cdot (e^{x^2-1}) \cdot (2x) = 2xe^{x^2-1}\cos(e^{x^2-1})$$

Ex. Find the equation of the tangent line to the function $y = x\sin(x^2+\frac{\pi}{2})$ at the origin?

SOLUTION:

You need the derivative … you'll need the product rule AND the chain rule on this one:

$$y' = (x)\left(\sin(x^2+\tfrac{\pi}{2})\right)' + (x)'\left(\sin(x^2+\tfrac{\pi}{2})\right)$$

$$y' = x\cos(x^2+\tfrac{\pi}{2}) \cdot (2x) + \sin(x^2+\tfrac{\pi}{2})$$

Evaluate at x = 0 → $y' = 0\cos(0+\frac{\pi}{2}) \cdot (0) + \sin(0+\frac{\pi}{2}) = 1$

The slope of the tangent line is 1 … the tangent line through the origin is the line $y = x$

Ex. Show that the function $y = Ae^{-2x} + Be^{3x}$ satisfies the differential equation $y'' - y' - 6y = 0$

SOLUTION:

You need to build the ingredients for the differential equation …

$y = Ae^{-2x} + Be^{3x}$ $\quad y' = -2Ae^{-2x} + 3Be^{3x}$ $\quad y'' = 4Ae^{-2x} + 9Be^{3x}$

$y'' - y' - 6y = 0$

→ substitute the different ingredients

$$(4Ae^{-2x} + 9Be^{3x}) - (-2Ae^{-2x} + 3Be^{3x}) - 6(Ae^{-2x} + Be^{3x}) = 0$$
$$4Ae^{-2x} + 9Be^{3x} + 2Ae^{-2x} - 3Be^{3x} - 6Ae^{-2x} - 6Be^{3x} = 0$$
$$\left(4Ae^{-2x} + 2Ae^{-2x} - 6Ae^{-2x}\right) + \left(9Be^{3x} - 3Be^{3x} - 6Be^{3x}\right) = 0$$

$0 = 0$ ← the solution checks out!