

MATH 1910

Implicit Differentiation

So far, all the relations between x and y have been **explicitly** solved for y as a function of x , as in $y = \sqrt{4 - x^2}$.

However, there are relations you'll encounter for which it is not possible to write y explicitly as a function of x .

An example of this would be $2x + e^y = 4y$, impossible to solve for y in this equation.

This is an example of an **implicit** relation between x and y .

You'll have to use **implicit differentiation** to find derivatives for implicit relations.

When you're using implicit differentiation, you differentiate both x 's and y 's the same way. Since we're assuming that y is a function of x , anytime we differentiate a y we multiply it with " dy/dx " using the chain rule.

Ex. Differentiate $4x^2 + y^2 = 16$.

SOLUTION:

$$\text{Differentiating we get} \quad \rightarrow \quad 8x + 2y \cdot \frac{dy}{dx} = 0$$

$$\text{Now, solve for } \frac{dy}{dx} \quad \rightarrow \quad \frac{dy}{dx} = \frac{-8x}{2y} = -\frac{4x}{y}.$$

$$\text{So, the implicit derivative of the ellipse } 4x^2 + y^2 = 16 \quad \text{is} \quad \frac{dy}{dx} = -\frac{4x}{y}.$$

Ex. Find the equation of the tangent line to the ellipse $4x^2 + y^2 = 16$ at the point $(\sqrt{3}, -2)$.

SOLUTION:

$$\text{We've already got the implicit derivative for this relation: } \frac{dy}{dx} = -\frac{4x}{y}$$

$$\text{The tangent line slope is found by plugging in the values from the point } \rightarrow \quad \frac{dy}{dx} = -\frac{4\sqrt{3}}{-2} = 2\sqrt{3}$$

$$\text{The equation of the tangent line is } y - (-2) = 2\sqrt{3}(x - \sqrt{3}) \quad \rightarrow \quad y = 2\sqrt{3} \cdot x - 8$$

Ex. Differentiate $y^3 + 3x^2y^2 + 5x^4 = 12$

SOLUTION: Differentiate left to right and apply the **product rule** to the middle term:

$$3y^2 \frac{dy}{dx} + 6xy^2 + 6x^2y \frac{dy}{dx} + 20x^3 = 0$$

Now, solve for $\frac{dy}{dx}$:

$$3y^2 \frac{dy}{dx} + 6x^2y \frac{dy}{dx} = -6xy^2 - 20x^3$$

$$(3y^2 + 6x^2y) \frac{dy}{dx} = -6xy^2 - 20x^3$$

$$\frac{dy}{dx} = \frac{-6xy^2 - 20x^3}{3y^2 + 6x^2y}$$

MATH 1910 Implicit Differentiation

Ex. Differentiate $x \sin y + \cos 2y = \cos y$.

SOLUTION: Differentiate left to right and apply the **product rule** to the first term:

$$1 \cdot \sin y + x \cdot \cos y \cdot \frac{dy}{dx} + (-2 \sin 2y \cdot \frac{dy}{dx}) = -\sin y \cdot \frac{dy}{dx} \quad \leftarrow \text{product and chain rule involved}$$

Now, solve for $\frac{dy}{dx}$:

$$x \cos y \frac{dy}{dx} - 2 \sin 2y \frac{dy}{dx} + \sin y \cdot \frac{dy}{dx} = -\sin y \quad \leftarrow \text{collect all the } \frac{dy}{dx} \text{ terms on the left}$$

$$(x \cos y - 2 \sin 2y + \sin y) \frac{dy}{dx} = -\sin y \quad \leftarrow \text{factor out the } \frac{dy}{dx} \text{ on the left}$$

$$\frac{dy}{dx} = \frac{-\sin y}{x \cos y - 2 \sin 2y + \sin y} \quad \leftarrow \text{divide to solve for } \frac{dy}{dx}$$

Ex. Show that the given curves are orthogonal (have perpendicular derivatives at their points of intersection)

$$x^2 - y^2 = 5, \quad 4x^2 + 9y^2 = 72$$

SOLUTION: Differentiate both curves implicitly

$$\text{For } x^2 - y^2 = 5: \quad 2x - 2y \frac{dy}{dx} = 0 \quad \rightarrow \quad -2y \frac{dy}{dx} = -2x \quad \rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$$

$$\text{For } 4x^2 + 9y^2 = 72: \quad 8x + 18y \frac{dy}{dx} = 0 \quad \rightarrow \quad 18y \frac{dy}{dx} = -8x \quad \rightarrow \quad \frac{dy}{dx} = -\frac{4x}{9y}$$

Now we need to find where the curves intersect.

Solving the system would be best done by substitution, make the substitution $x^2 = y^2 + 5$ into the other equation $4x^2 + 9y^2 = 72$:

$$4x^2 + 9y^2 = 72 \quad \rightarrow \quad 4(y^2 + 5) + 9y^2 = 72 \quad \rightarrow \quad 13y^2 = 52 \quad \rightarrow \quad y^2 = 4 \quad \rightarrow \quad y = \pm 2$$

Now, since $x^2 = y^2 + 5$, we get the x solutions: $x^2 = (\pm 2)^2 + 5 = 9 \quad \rightarrow \quad x = \pm 3$

These two curves intersect at 4 points: $(3, 2)$, $(-3, 2)$, $(3, -2)$, $(-3, -2)$

Plugging each point pair into the derivatives gives us the following slopes:

	(3,2)	(-3,2)	(3,-2)	(-3,-2)
$x^2 - y^2 = 5$	$\frac{dy}{dx} = \frac{3}{2}$	$\frac{dy}{dx} = -\frac{3}{2}$	$\frac{dy}{dx} = -\frac{3}{2}$	$\frac{dy}{dx} = \frac{3}{2}$
$4x^2 + 9y^2 = 72$	$\frac{dy}{dx} = -\frac{2}{3}$	$\frac{dy}{dx} = \frac{2}{3}$	$\frac{dy}{dx} = \frac{2}{3}$	$\frac{dy}{dx} = -\frac{2}{3}$

So these curves are perpendicular, or orthogonal, at their points of intersection because their derivative values at those points are opposite reciprocals.

MATH 1910

Implicit Differentiation

Derivative of the Natural Logarithm function

This derivative will not be used until section 3.7 but we will use implicit differentiation to derive it.

Start with $y = \ln x$, and rewrite it as $e^y = x$

Now, use implicit differentiation on $e^y = x$

$$e^y \frac{dy}{dx} = 1 \quad \leftarrow \text{solve for } \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{e^y} \quad \leftarrow \text{now make the substitution } y = \ln x$$

$$\frac{dy}{dx} = \frac{1}{e^{\ln x}} \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \quad \leftarrow \text{the cancellation property causes } e^{\ln x} = x$$

So our final function derivative for this chapter is $\frac{d}{dx}(\ln x) = \frac{1}{x}$

The chain rule version of this derivative is $\frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$