MATH 1910
Section 3.7 Derivatives of Logarithmic Functions

We've established the **derivative of the natural logarithm**:

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

The **chain rule** version of this derivative looks like

$$\boxed{\frac{d}{dx}\left[\ln(f(x))\right] = \frac{1}{f(x)} \cdot f'(x)}$$ ← this is the one you'll use most often

Ex. Differentiate $y = \ln(2x^4 - x)$
SOLUTION: Using the chain rule version of the natural log's derivative we get

$$y' = \frac{1}{2x^4 - x} \cdot (8x^3 - 1) = \frac{8x^3 - 1}{2x^4 - x}$$

Ex. Differentiate $f(x) = \log_3 x$
SOLUTION:

First you need to use the base change formula for logarithms to rewrite $\log_3 x = \dfrac{\ln x}{\ln 3}$

Now we can differentiate $f(x) = \dfrac{\ln x}{\ln 3}$, (remember: $\ln 3$ is just a constant value, the quotient rule is not necessary).

$$f(x) = \frac{1}{\ln 3} \cdot \ln x \qquad f'(x) = \frac{1}{\ln 3} \cdot \left(\frac{1}{x}\right) = \frac{1}{x \ln 3}$$ ← done!

Sometimes it might be to your advantage to employ the use of the **laws of logarithms** *before* differentiating, it could make the problem a WHOLE lot easier. Here they are in case you've forgotten them:

I. $\ln(uv) = \ln u + \ln v$ II. $\ln\left(\frac{u}{v}\right) = \ln u - \ln v$ III. $\ln u^n = n \ln u$

Ex. Differentiate $y = \ln\left(x^3 \cdot \sqrt{x^2 + 1}\right)$

SOLUTION:

Using the laws of logarithms you can rewrite as $y = \ln(x^3) + \ln(\sqrt{x^2 + 1})$ using law I.

Then the powers can be simplified using law III and we get $y = 3\ln(x) + \frac{1}{2}\ln(x^2 + 1)$.

Now, we can differentiate according to the chain rule → $\dfrac{d}{dx}\left[\ln(f(x))\right] = \dfrac{1}{f(x)} \cdot f'(x)$.

$$y' = 3 \cdot \frac{1}{x} + \frac{1}{2} \cdot \left(\frac{1}{x^2 + 1}\right) \cdot 2x \rightarrow \; y' = \frac{3}{x} + \frac{x}{x^2 + 1}$$ and we're done!

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Ex. Differentiate $f(x) = \ln(\csc x - \cot x)$.
SOLUTION: Chain rule …

$$f(x) = \ln(\csc x - \cot x) \quad \rightarrow \quad f'(x) = \frac{1}{\csc x - \cot x} \cdot \left(-\csc x \cot x + \csc^2 x\right)$$

$$f'(x) = \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x}$$

$$f'(x) = \frac{\csc x \cdot (\csc x - \cot x)}{\csc x - \cot x} \quad \rightarrow \quad f'(x) = \csc x$$

(this suggests that $f(x) = \ln(\csc x - \cot x)$ is an antiderivative for the cosecant function)

LOGARITHMIC DIFFERENTIATION:

1. Take the natural logarithm of both sides of an equation $y = f(x)$ and use the laws of logs to simplify.
2. Differentiate implicitly with respect to *x*.
3. Solve the resulting equation for dy/dx and replace any single *y*s with $f(x)$.

Ex. Differentiate the function $f(x) = x^x$.
SOLUTION:

Step 1: take the natural log of both sides of the equation $y = x^x$ → $\ln y = \ln x^x$
Now simplify using the laws of logs: $\ln y = x \ln x$

Step 2: Differentiate implicitly on both sides: $\frac{1}{y} \cdot \frac{dy}{dx} = (x)(\ln x)' + (x)'(\ln x)$ (product rule on RHS)

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \ln x = 1 + \ln x \quad \rightarrow \text{ so } \frac{1}{y} \cdot \frac{dy}{dx} = 1 + \ln x$$

Step 3: Solve the equation for $\frac{dy}{dx}$ and replace any *y*s with $f(x)$: $\frac{dy}{dx} = y \cdot (1 + \ln x)$ → replace y with x^x because that's the way y was defined: $\frac{dy}{dx} = x^x \cdot (1 + \ln x)$ → done!

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Ex. Differentiate $y=\sqrt{\dfrac{x+1}{x-1}}$ using logarithmic differentiation.

SOLUTION: take the natural log of both sides of the equation and simplify the expression:

$\ln y=\ln\sqrt{\dfrac{x+1}{x-1}}$ → (rule III) $\ln y=\dfrac{1}{2}\ln\dfrac{x+1}{x-1}$ → (rule II) $\ln y=\dfrac{1}{2}\ln(x+1)-\dfrac{1}{2}\ln(x-1)$.

Now differentiate implicitly → $\dfrac{1}{y}\cdot\dfrac{dy}{dx}=\dfrac{1}{2}\cdot\dfrac{1}{x+1}-\dfrac{1}{2}\cdot\dfrac{1}{x-1}$

Now, solve for $\dfrac{dy}{dx}$ and replace y → $\dfrac{dy}{dx}=y\cdot\left(\dfrac{1}{2(x+1)}-\dfrac{1}{2(x-1)}\right)=\sqrt{\dfrac{x+1}{x-1}}\cdot\left(\dfrac{1}{2(x+1)}-\dfrac{1}{2(x-1)}\right)$

Ex. Differentiate the function $y=\dfrac{x^{2/3}(4x^2+1)^{10}}{\sqrt{\sin x}}$ using logarithmic differentiation.

SOLUTION: take the "LN" of both sides of the original function

$\ln y=\ln\left(\dfrac{x^{2/3}(4x^2+1)^{10}}{\sqrt{\sin x}}\right)$ → use properties of logarithms to simplify the right hand side

$$\ln y=\ln(x)^{2/3}+\ln(4x^2+1)^{10}-\ln(\sin x)^{1/2}$$

$$\ln y=\tfrac{2}{3}\ln(x)+10\ln(4x^2+1)-\tfrac{1}{2}\ln(\sin x)$$

NOW … differentiate both sides implicitly

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{2}{3}\cdot\left(\frac{1}{x}\right)+10\cdot\left(\frac{1}{4x^2+1}\right)\cdot(8x)-\frac{1}{2}\cdot\left(\frac{1}{\sin x}\right)\cdot\cos x$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{2}{3x}+\frac{80x}{4x^2+1}-\frac{1}{2}\cdot\left(\frac{\cos x}{\sin x}\right)$$

Now, solve for $\dfrac{dy}{dx}$ and replace y: $\dfrac{dy}{dx}=y\cdot\left(\dfrac{2}{3x}+\dfrac{80x}{4x^2+1}-\dfrac{\cot x}{2}\right)$

$$=\left(\frac{x^{2/3}(4x^2+1)^{10}}{\sqrt{\sin x}}\right)\cdot\left(\frac{2}{3x}+\frac{80x}{4x^2+1}-\frac{\cot x}{2}\right)$$

Ex. Differentiate $y=2^{x-1}$

SOLUTION: Use logarithmic differentiation to find y':

$\ln y=\ln 2^{x-1}$ → $\ln y=(x-1)\ln 2$

Now differentiate both sides implicitly: → $\dfrac{1}{y}\dfrac{dy}{dx}=\ln 2$

Now solve for $\dfrac{dy}{dx}$ and replace y: → $\dfrac{dy}{dx}=y\ln 2$ → $\dfrac{dy}{dx}=2^{x-1}\ln 2$

Homework: Same as on policy sheet, ignore the absolute value in problem #15