APPLICATIONS OF DIFFERENTIATION

4.3 DERIVATIVES AND THE SHAPES OF CURVES

A. 1. $f(x) = x\ln(x^2)$ — D: $(-\infty, 0) \cup (0, \infty)$

2. $f'(x) = \ln(x^2) + x\left(\dfrac{2x}{x^2}\right) = \ln(x^2) + 2$

$\ln(x^2) + 2 = 0 \Rightarrow \ln(x^2) = -2 \Rightarrow e^{-2} = x^2 \Rightarrow x = \pm\dfrac{1}{e} \approx \pm.368$ — CN: $\pm\dfrac{1}{e}$

Horizontal tangents at x = $\pm\dfrac{1}{e}$

3. First derivative sign chart for increasing/decreasing intervals

f'	+ + +		- - -		- - -		+ + +
f	↗	-.368	↘	0	↘	.368	↗
test	-1		-.2		.2		1

$f'(-1) = \ln(-1)^2 + 2 = 2 > 0$

$f'(-.2) = \ln(-.2)^2 + 2 \approx -1.219 < 0$

$f'(.2) = \ln(.2)^2 + 2 \approx -1.219 < 0$

$f'(1) = \ln(1)^2 + 2 > 0$

↗ on $\left(-\infty, -\dfrac{1}{e}\right) \cup \left(\dfrac{1}{e}, \infty\right)$

↘ on $\left(-\dfrac{1}{e}, 0\right) \cup \left(0, \dfrac{1}{e}\right)$

R: $(-\infty, 0) \cup (0, \infty)$

4. $f\left(-\dfrac{1}{e}\right) = -\dfrac{1}{e}\ln\left(-\dfrac{1}{e}\right)^2 \approx .736$ — Local max: .736 at x = -.368

$f\left(\dfrac{1}{e}\right) = \dfrac{1}{e}\ln\left(\dfrac{1}{e}\right)^2 \approx -.736$ — Local min: -.736 at x = .368

No absolute extrema

5. f(0) is undefined — No y-intercepts

$0 = x\ln(x^2) \Rightarrow x = \pm 1$ — x-intercepts: $(-1, 0)$, $(1, 0)$

6. $f'(x) = \ln(x^2) + 2 \Rightarrow f'' = \dfrac{2x}{x^2} = \dfrac{2}{x}$

f'' cannot = 0
f'' is undefined if x = 0. There cannot be an inflection point here, [not in the domain of the function] but it is still a point of interest!

7. Second derivative sign chart for concavity

f''	- - -		+ + +
f	∩	0	∪
test	-2		2

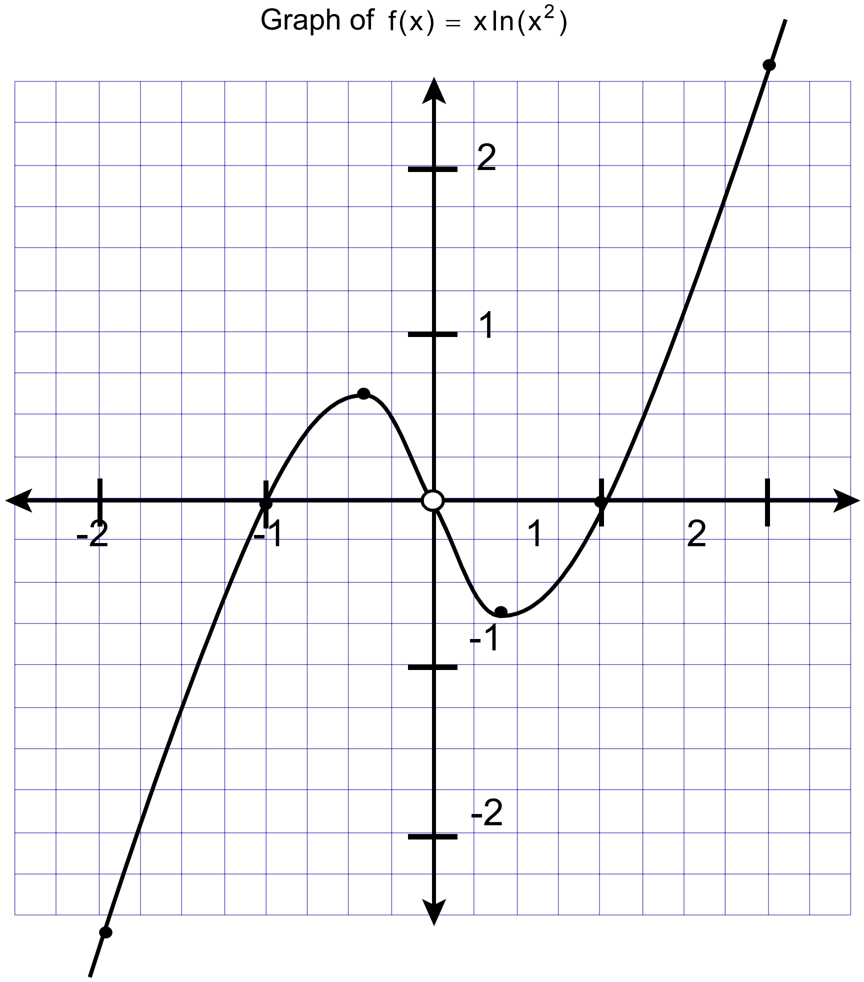
$f''(-2) = -\dfrac{2}{2} < 0$ and $f''(2) = \dfrac{2}{2} > 0$

∩ on $(-\infty, 0)$

∪ on $(0, \infty)$

No inflection points

Graph of $f(x) = x\ln(x^2)$



Additional points: f(-2) = -2.773, f(2) = 2.773

B. 1. $f(x) = x^{2/3}(5 + x)$ D: $(-\infty, 0) \cup (0, \infty)$

2. $f'(x) = (5 + x)\frac{2}{3}x^{-1/3} + x^{2/3} = \frac{5}{3}\left(\frac{2 + x}{x^{1/3}}\right)$

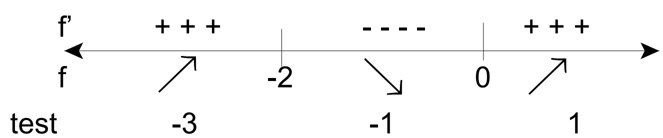
$f'(x) = \frac{5}{3}\left(\frac{2 + x}{x^{1/3}}\right) = 0 \Rightarrow x = -2$

$f'(x) = \frac{5}{3}\left(\frac{2 + x}{x^{1/3}}\right)$ is undefined if x = 0 CN: x = -2, 0

Horizontal tangent at x = -2

There is no horizontal tangent at x = 0, because there is no derivative there.

3. First derivative sign chart for increasing/decreasing intervals



↗ on $(-\infty, -2) \cup (0, \infty)$

↘ on (-2, 0)

$f'(-3) = \frac{5}{3}\left(\frac{2 - 3}{(-3)^{1/3}}\right) > 0$

$f'(-1) = \frac{5}{3}\left(\frac{2 - 1}{(-1)^{1/3}}\right) < 0$, $f'(1) = \frac{5}{3}\left(\frac{2 + 1}{(1)^{1/3}}\right) > 0$ R: $(-\infty, \infty)$

4. $f(-2) = (-2)^{2/3}(5 - 2) \approx 4.76$ Local max: 4.76 at x = -2

$f(0) = 0$ Local min: 0 at x = 0

No absolute extrema

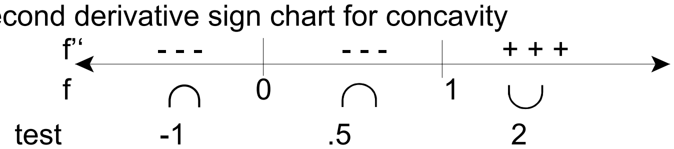
5. f(0) = 0 y-intercept: (0, 0)

$0 = x^{2/3}(5 + x) \Rightarrow x = 0$ or $x = -5$ x-intercepts: $(-5, 0)$, $(0, 0)$

6. $f'(x) = \frac{5}{3}(2x^{-1/3} + x^{2/3}) \Rightarrow f''(x) = \frac{5}{3}\left(-\frac{2}{3}x^{-4/3} + \frac{2}{3}x^{-1/3}\right) = \frac{10}{9}\left(\frac{x - 1}{x^{4/3}}\right)$

f'' = 0 if x = 1; f'' is undefined if x = 0 Possible inflection points at x = 0, 1

7. Second derivative sign chart for concavity



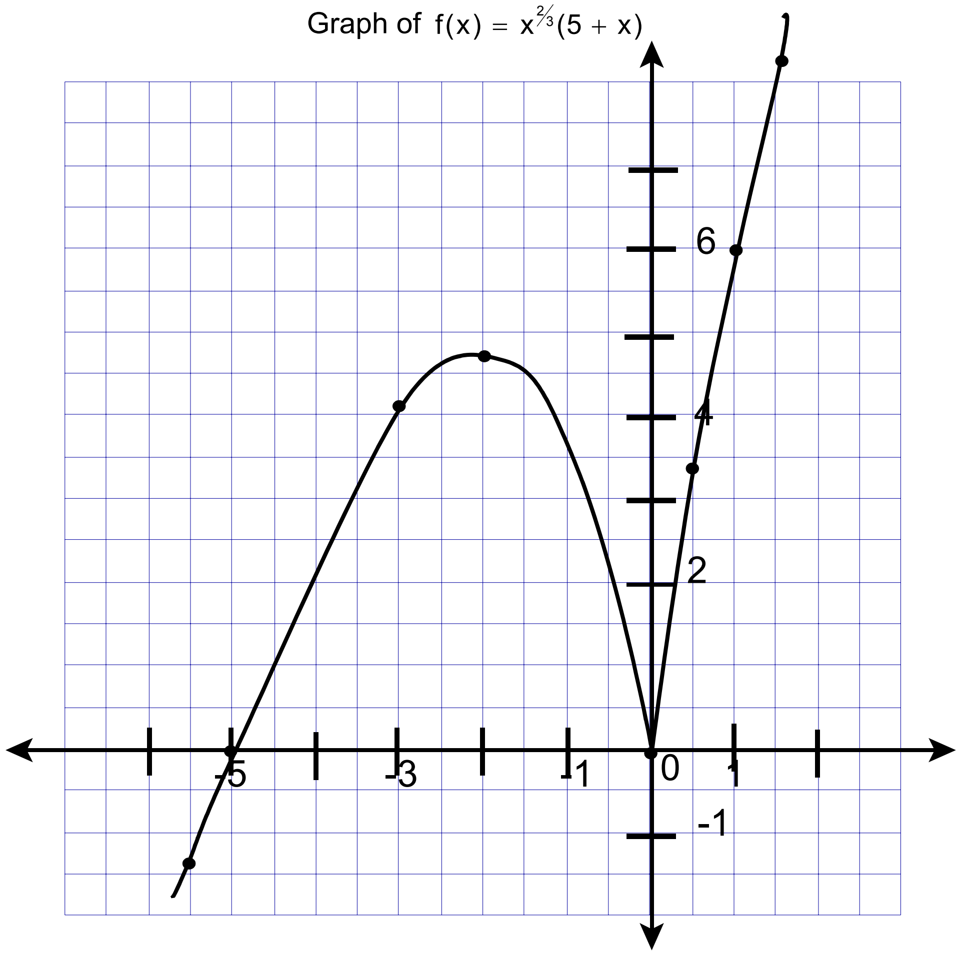
∩ on $(-\infty, 0)$ and (0, 1)

∪ on $(0, \infty)$

$f''(-1) = \frac{10}{9}\left(\frac{-2}{1}\right) < 0$, $f''(.5) = \frac{10}{9}\left(\frac{-.5}{\sqrt[3]{.0525}}\right) < 0$, $f''(2) = \frac{10}{9}\left(\frac{1}{\sqrt[3]{16}}\right) > 0$

f(0) is defined, but f'(0) is not. There is a cusp at f(0). Inflection point is (1, 6)

Graph of $f(x) = x^{2/3}(5 + x)$



Additional points: f(-5.5) = -1.56, f(-3) = -4.16, f(.5) = 3.46, f(1.5) = 8.52

8. Second derivative test

$f''(-2) < 0 \Rightarrow \cap \Rightarrow$ local max at x = -2

$f''(0)$ is undefined, which does not give any information

C. 1. $f(x) = \dfrac{4 + x^2}{4 - x^2}$

D: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Asymptotes: x = 2, x = -2

2. $f' = \dfrac{16x}{(4 - x^2)^2}$

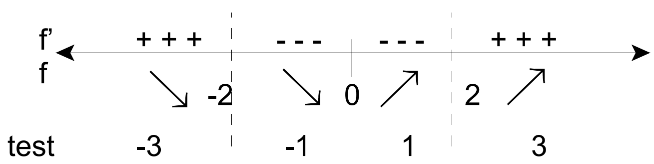
$f' = \dfrac{16x}{(4 - x^2)^2} = 0 \Rightarrow x = 0$

CN: 0

Horizontal tangent at x = 0

f'(x) is undefined if x = ± 2. Although these are not critical numbers [not in domain], they are still points of interest!

3. First derivative sign chart for increasing/decreasing intervals



↘ on $(-\infty, -2) \cup (-2, 0)$

↗ on $(0, 2) \cup (2, \infty)$

$f'(-3) = \dfrac{-48}{(-5)^2} < 0$, $f'(-1) = \dfrac{-16}{(3)^2} < 0$

$f'(1) = \dfrac{16}{(3)^2} > 0$, $f'(3) = \dfrac{48}{(-5)^2} > 0$

4. $f(0) = 1$

Local min: 1 at x = 0

No absolute extrema

5. f(0) = 1

y-intercepts: (0, 1)

$f(x) = \dfrac{4 + x^2}{4 - x^2} = 0 \Rightarrow 4 + x^2 = 0 \Rightarrow x^2 = -4$

No x-intercepts

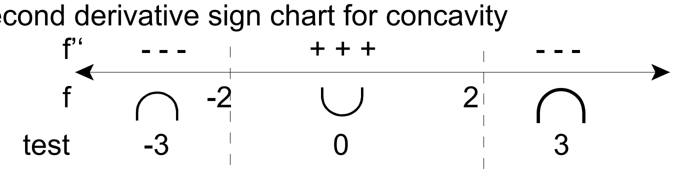
6. $f' = \dfrac{16x}{(4 - x^2)^2} \Rightarrow f'' = \dfrac{16(4 + 3x^2)}{(4 - x^2)^3}$

f'' cannot = 0

No inflection points

f'' is undefined if x = ± 2 . There cannot be an inflection points here, [not in the domain of the function] but these numbers are still of interest!

7. Second derivative sign chart for concavity



∩ on $(-\infty, -2) \cup (2, \infty)$

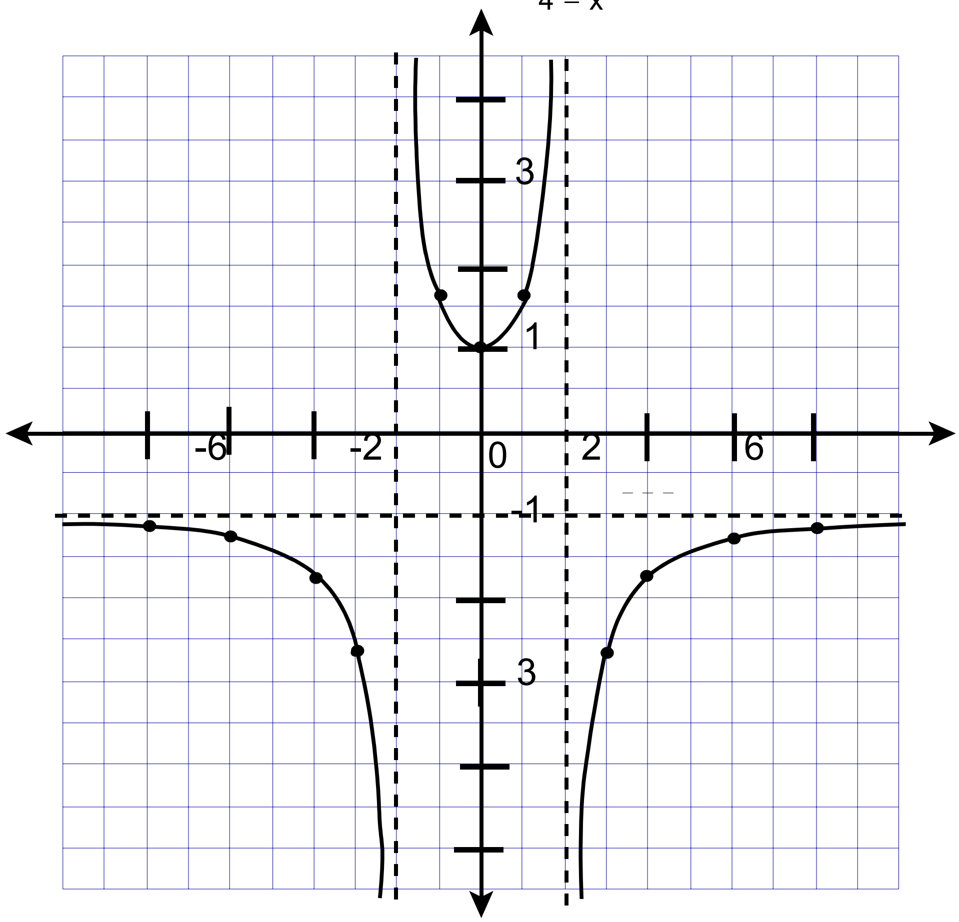
∪ on $(-2, 2)$

$f''(-3) = \dfrac{16(4 + 27)}{(4 - 9)^3} < 0$, $f''(0) = \dfrac{16(4)}{(4)^3} > 0$, $f''(3) = \dfrac{16(4 + 27)}{(4 - 9)^3} < 0$

8. $\lim\limits_{x \to \infty} \dfrac{4 + x^2}{4 - x^2} = \lim\limits_{x \to \infty} \dfrac{\frac{4}{x^2} + 1}{\frac{4}{x^2} - 1} = -1 \Rightarrow y = -1$ is a horizontal asymptote

R: $(-\infty, -1) \cup [0, \infty)$

Graph of $f(x) = \frac{4 + x^2}{4 - x^2}$



Additional points: f(-3) = -2.6, f(-4) = -1.7, f(-6) = -1.25, f(-8) = -1.1, f(3) = -2.6, f(4) = -1.7, f(6) = -1.25, f(8) = -1.1, f(-1) = 1.7, f(1) = 1.7