APPLICATIONS OF DIFFERENTIATION

4.5 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

I. Necessary conditions for applying l'Hospital's Rule

- A. f and g are both differentiable ***near*** a (except possibly at a)
- B. $g'(x) \neq 0$ near a (except possibly at a)
- C. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$; these are called ***indeterminate forms***
- D. l'Hospital's Rule also applies if a is replaced by $\pm\infty$

II. Result of applying l'Hospital's Rule: $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

III. You must show that the rule applies before using it. Always simplify before applying the rule another time!

IV. Indeterminate form $\frac{\infty}{\infty}$: $\lim_{x\to\infty}\frac{4x^2-3x+6}{5-2x^2}$

- A. Previous method: $\lim_{x\to\infty}\frac{4x^2-3x+6}{5-2x^2} = \lim_{x\to\infty}\frac{\frac{4x^2}{x^2}-\frac{3x}{x^2}+\frac{6}{x^2}}{\frac{5}{x^2}-\frac{2x^2}{x^2}} = \lim_{x\to\infty}\frac{4-\frac{3}{x}+\frac{6}{x^2}}{\frac{5}{x^2}-2} = -2$
- B. l'Hospital's Rule applies because $\lim_{x\to\infty}\frac{4x^2-3x+6}{5-2x^2} = \frac{\infty}{\infty}$
  1. $\lim_{x\to\infty}\frac{4x^2-3x+6}{5-2x^2} \overset{H}{=} \lim_{x\to\infty}\frac{8x-3}{-4x} = \frac{\infty}{\infty}$
  2. $\lim_{x\to\infty}\frac{8x-3}{-4x} \overset{H}{=} \lim_{x\to\infty}\frac{8}{-4} = -2$

V. Indeterminate form $\frac{0}{0}$: $\lim_{x\to 0}\frac{\sin x}{x+x^2}$

- A. $\lim_{x\to 0}\frac{\sin x}{x+x^2} = \frac{0}{0}$, so l'Hospital's Rule applies
- B. $\lim_{x\to 0}\frac{\sin x}{x+x^2} \overset{H}{=} \lim_{x\to 0}\frac{\cos x}{1+2x} = \frac{1}{1} = 1$

VI. Repeated application of l'Hospital's Rule: $\lim_{x\to\infty}\frac{e^x}{x^3} = \frac{\infty}{\infty}$

- A. $\lim_{x\to\infty}\frac{e^x}{x^3} \overset{H}{=} \lim_{x\to\infty}\frac{e^x}{3x^2} = \frac{\infty}{\infty}$
- B. $\lim_{x\to\infty}\frac{e^x}{3x^2} \overset{H}{=} \lim_{x\to\infty}\frac{e^x}{6x} = \frac{\infty}{\infty}$

C. $\lim\limits_{x\to\infty} \frac{e^x}{6x} \overset{H}{=} \lim\limits_{x\to\infty} \frac{e^x}{6} = \infty$

VII. Indeterminate product of type $(\pm\infty)(0)$: $\lim\limits_{x\to-\infty} xe^x$

A. $\lim\limits_{x\to-\infty} xe^x = (-\infty)(0)$

B. Rewrite to obtain an indeterminate form: $\lim\limits_{x\to-\infty} xe^x = \lim\limits_{x\to-\infty} \frac{x}{e^{-x}} = \frac{-\infty}{\infty}$

C. $\lim\limits_{x\to-\infty} \frac{x}{e^{-x}} \overset{H}{=} \lim\limits_{x\to-\infty} \frac{1}{-e^{-x}} = 0$

VIII. Indeterminate difference of type $\infty - \infty$: $\lim\limits_{x\to 0} (\csc x - \cot x)$

A. $\lim\limits_{x\to 0} (\csc x - \cot x) = \infty - \infty$

B. Rewrite to obtain an indeterminate form:

$$\lim_{x\to 0} (\csc x - \cot x) = \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0} \left(\frac{1-\cos x}{\sin x}\right) = \frac{0}{0}$$

C. $\lim\limits_{x\to 0} \left(\frac{1-\cos x}{\sin x}\right) \overset{H}{=} \lim\limits_{x\to 0} \frac{\sin x}{\cos x} = 0$

IX. Indeterminate power types include 0^0, ∞^0, and 1^∞: $\lim\limits_{x\to\infty} \left(1-\frac{a}{x}\right)^{bx}$

A. $\lim\limits_{x\to\infty} \left(1-\frac{a}{x}\right)^{bx} = 1^\infty$

B. Use natural log when indeterminate powers are involved: Let $y = \left(1-\frac{a}{x}\right)^{bx}$

C. $\ln y = (bx)\ln\left(1-\frac{a}{x}\right)$

D. $\lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} (bx)\ln\left(1-\frac{a}{x}\right) = (\infty)(0)$

E. $\lim\limits_{x\to\infty} (bx)\ln\left(1-\frac{a}{x}\right) = \lim\limits_{x\to\infty} \frac{(b)\ln\left(1-\frac{a}{x}\right)}{1/x} = \frac{0}{0}$

F. $\lim\limits_{x\to\infty} \frac{(b)\ln\left(1-\frac{a}{x}\right)}{1/x} \overset{H}{=} \lim\limits_{x\to\infty} \frac{\left[b\left(\frac{1}{1-a/x}\right)\left(-\frac{a}{x^2}\right)\right]}{-1/x^2}\left(\frac{-x^2}{-x^2}\right) = \lim\limits_{x\to\infty} \frac{ab}{1+a/x}$

G. Since $\lim\limits_{x\to\infty} \frac{a}{x} = 0$, $\lim\limits_{x\to\infty} \frac{ab}{1+a/x} = ab$

H. This is the $\lim\limits_{x\to\infty} \ln y$, not the limit of y itself!

I. $\lim_{x \to \infty} \ln y = ab \Rightarrow \lim_{x \to \infty} y = e^{ab}$