10.2 HW

1. a) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots+\frac{1}{3^n} \to a_n=\frac{1}{3^n}$

b.) $\frac{1}{1}+\frac{5}{2}+\frac{25}{4}+\frac{125}{8}+\cdots+\frac{5^{n-1}}{2^{n-1}} \to a_n=\left(\frac{5}{2}\right)^{n-1}$

c.) $\frac{1}{1}-\frac{2^2}{2\cdot 1}+\frac{3^3}{3\cdot 2\cdot 1}-\frac{4^3}{4\cdot 3\cdot 2\cdot 1}+\cdots+\frac{(-1)^{n+1}n^n}{n!} \to a_n=\frac{(-1)^{n+1}n^n}{n!}$

d.) $\frac{2}{1^2+1}+\frac{1}{2^2+1}+\frac{2}{3^2+1}+\frac{1}{4^2+1}+\cdots$ must use piece-wise

$$a_n=\begin{cases}\frac{2}{n^2+1}, & n \text{ odd}\\ \frac{1}{n^2+1}, & n \text{ even}\end{cases}$$

3. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$

$S_2 = 1+\frac{1}{4}=\frac{5}{4}$

$S_4 = S_2+\frac{1}{3^2}+\frac{1}{4^2}=\frac{5}{4}+\frac{1}{9}+\frac{1}{16}=\frac{205}{144}$

$S_6 = S_4+\frac{1}{5^2}+\frac{1}{6^2}=\frac{205}{144}+\frac{1}{25}+\frac{1}{36}=\frac{5369}{3600}$

5. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}+\cdots$

$S_2=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$

$S_4=S_2+\frac{1}{12}+\frac{1}{20}=\frac{4}{5}$

$S_6=S_4+\frac{1}{30}+\frac{1}{42}=\frac{6}{7}$

13. $S=\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$ $\qquad \frac{1}{4n^2-1}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$

$S_3=\frac{1}{2}\left[\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\frac{1}{7}\right)\right]=\frac{1}{2}\left(1-\frac{1}{7}\right)=\frac{3}{7}$

$S_4=\frac{1}{2}\left[\left(\frac{1}{7}-\frac{1}{9}\right)\right]+S_3 = \frac{1}{2}\left(1-\frac{1}{9}\right)=\frac{4}{9}$

$S_5=\frac{1}{2}\left(\frac{1}{9}-\frac{1}{11}\right)+S_4 = \frac{1}{2}\left(1-\frac{1}{11}\right)=\frac{5}{11}$

$S_n=\frac{1}{2}\left(1-\frac{1}{n}\right) \to$ as $n\to\infty$ $S_n=\lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{n}\right)=\frac{1}{2}$

15. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots \frac{1}{(2n-1)(2n+1)}$

so $S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ (same as #13) $= \frac{1}{2}$

17. $\sum_{n=1}^{\infty} \frac{n}{10n+12}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{10n+12} = \frac{1}{10} \therefore \sum_{n=1}^{\infty} \frac{n}{10n+12}$ diverges.

19. $\frac{0}{1} - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots \frac{(-1)^{n+1}(n-1)}{n}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)}{n}$ diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1}(n-1)}{n} \neq 0.$

21. $\cos(\frac{1}{2}) + \cos(\frac{1}{3}) + \cos(\frac{1}{4}) + \cdots$

$\sum_{n=1}^{\infty} \cos(\frac{1}{n+1})$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(\frac{1}{n+1}) = 1 \therefore \sum \cos(\frac{1}{n+1})$ is divergent.

23. $\frac{1}{1} + \frac{1}{8} + \frac{1}{8^2} + \cdots + \left(\frac{1}{8}\right)^n \rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n = \frac{1}{1-\frac{1}{8}} = \frac{8}{8-1} = \frac{8}{7}$

25. $\sum_{n=3}^{\infty} \left(\frac{3}{11}\right)^{-n} = \sum_{n=3}^{\infty} \left(\frac{11}{3}\right)^n$ $\lim_{n\to\infty} \left(\frac{11}{3}\right)^n \neq 0 \therefore \sum_{n=3}^{\infty} \left(\frac{11}{3}\right)^n$ diverges

27. $\sum_{n=-4}^{\infty} \left(-\frac{4}{9}\right)^n = \frac{\left(-\frac{4}{9}\right)^{-4}}{1-\left(-\frac{4}{9}\right)} = \frac{\frac{6561}{256}}{\frac{13}{9}} = \frac{6561}{256} \cdot \frac{9}{13} = \frac{59049}{3328}$

31. ~~29.~~ $\sum_{n=0}^{\infty} \frac{8+2^n}{5^n} = \sum_{n=0}^{\infty} 8\left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

$= \frac{8}{1-\frac{1}{5}} + \frac{1}{1-\frac{2}{5}} = \frac{40}{5-1} + \frac{5}{5-2} = 10 + \frac{5}{3} = \frac{35}{3}$

29. $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$

32. $$\sum_{n=0}^{\infty} \frac{3(-2)^n - 5^n}{8^n} = \sum_{n=0}^{\infty} 3\left(\frac{-1}{4}\right)^n - \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n$$

$$= \frac{3}{1-\left(-\frac{1}{4}\right)} - \frac{1}{1-\frac{5}{8}} = \frac{12}{4+1} - \frac{8}{8-5} = \frac{12}{5} - \frac{8}{3} = \frac{-4}{15}$$

33. $$5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \dots = \sum_{n=0}^{\infty} 5\left(\frac{-1}{4}\right)^n = \frac{5}{1-\left(-\frac{1}{4}\right)} = \frac{20}{4+1} = 4$$