10.5 HW

1. $\sum_{n=1}^{\infty} \frac{1}{5^n}$

$L = \lim_{n\to\infty} \left| \frac{1}{5^{n+1}} \cdot \frac{5^n}{1} \right| = \lim_{n\to\infty} \left| \frac{5^n}{5^{n+1}} \right| = \lim_{n\to\infty} \frac{1}{5} = \frac{1}{5}$

$\therefore \sum_{n=1}^{\infty} \frac{1}{5^n}$ converges absolutely.

3. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

$L = \lim_{n\to\infty} \left| \frac{1}{(n+1)^{n+1}} \cdot \frac{n^n}{1} \right| = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{n^{n+1}+\cdots}$

$L = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \therefore \sum_{n=1}^{\infty} \frac{1}{n^n}$ converges absolutely.

5. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$L = \lim_{n\to\infty} \left| \frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{n^2+1}{(n+1)^2+1} = 1 \cdot 1 = 1$

$\therefore$ Ratio Test is inconclusive.

7. $\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$

$L = \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^{100}} \cdot \frac{n^{100}}{2^n} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n^{100}}{(n+1)^{100}} = 2 \cdot 1 = 2$

$\therefore \sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$ diverges.

9. $\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$

$L = \lim_{n\to\infty} \left| \frac{10^{n+1}}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{10^n} \right| = \lim_{n\to\infty} \frac{10^{n+1}}{10^n} \cdot \frac{2^{n^2}}{2^{(n+1)^2}}$

$L = \lim_{n\to\infty} 10 \cdot \frac{2^{n^2}}{2^{n^2+2n+1}} = \lim_{n\to\infty} \frac{10}{2^{2n+1}} = 0$

$\therefore \sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$ converges absolutely.

11. $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$

$$L = \lim_{n\to\infty} \left| \frac{e^{n+1}}{n^{n+1}} \cdot \frac{n^n}{e^n} \right| = \lim_{n\to\infty} \left| \frac{e^{n+1}}{e^n} \cdot \frac{n^n}{n^{n+1}} \right| = \lim_{n\to\infty} \frac{e}{n} = 0$$

$\therefore \sum_{n=1}^{\infty} \frac{e^n}{n^n}$ converges absolutely.

13. $\sum_{n=0}^{\infty} \frac{n!}{6^n}$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!}{6^{n+1}} \cdot \frac{6^n}{n!} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{6^n}{6^{n+1}} = \lim_{n\to\infty} \frac{n+1}{6} = \infty$$

$\therefore \sum_{n=0}^{\infty} \frac{n!}{6^n}$ diverges.

15. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$$L = \lim_{n\to\infty} \left| \frac{1}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{1} \right| = \lim_{n\to\infty} \frac{n \ln n}{(n+1)\ln(n+1)} = 1$$

Ratio Test is inconclusive.

17. $\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{n^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{(2n+1)!}{(2n+3)!} \right|$$

$$L = \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0$$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$ converges absolutely.

35. $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$L = \lim_{n\to\infty} \left| \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} \right| = \lim_{n\to\infty} \frac{n^p}{n^p + \cdots} = 1$$ $\therefore$ RT is inconclusive.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \therefore \sum_{n=1}^{\infty} \frac{1}{n^n} \text{ converges absolutely.}$$

39. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$

$$L = \lim_{k\to\infty} \sqrt[k]{\left(\frac{k}{3k+1}\right)^k} = \lim_{k\to\infty} \frac{k}{3k+1} = \frac{1}{3} \quad \therefore \sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k \text{ converges. absolutely}$$

41. $\displaystyle\sum_{n=4}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2} = \sum_{n=4}^{\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \quad \therefore \sum_{n=4}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2} \text{ converges absolutely.}$$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n$

Since both $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n$ and $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n$ are convergent geometric series $(r<1)$, $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{7^n}$ also converges.

45. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

$$R.T. \rightarrow L = \lim_{n\to\infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{5^n}{5^{n+1}} = \frac{1}{5}$$

$$\therefore \sum_{n=1}^{\infty} \frac{n^3}{5^n} \text{ converges absolutely.}$$

47. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-n^2}}$ LCT with convergent $\frac{1}{n^{3/2}}$

$$L = \lim_{n\to\infty} \frac{1}{\sqrt{n^3-n^2}} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3-n^2}} = 1$$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-n^2}} \text{ also converges.}$$

49. $\sum_{n=1}^{\infty} n^{-.8} = \sum_{n=1}^{\infty} \frac{1}{n^{.8}}$

This is a divergent p-series ($p = .8 \le 1$).

51. $\sum_{n=1}^{\infty} 4^{-2n+1} = \sum_{n=1}^{\infty} \frac{4}{4^{2n}} = \sum_{n=1}^{\infty} 4\left(\frac{1}{16}\right)^n$

This is a convergent geometric series ($r = \frac{1}{16} < 1$).