1. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{2} \right| < 1$$

$$-1 < \left| \frac{x}{2} \right| < 1 \rightarrow -2 < x < 2$$

check endpoints...

$x = -2 \rightarrow \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$ this diverges since $\lim_{n\to\infty} a_n \neq 0$.

$x = 2 \rightarrow \sum_{n=0}^{\infty} \frac{(2)^n}{2^n} = \sum_{n=0}^{\infty} 1$ again this diverges since $\lim_{n\to\infty} a_n \neq 0$.

Interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ is $x = (-2, 2)$.

3. a.) $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3} \right| < 1$$

$$-1 < \frac{x}{3} < 1 \rightarrow -3 < x < 3$$

check endpoints...

$x = -3 \rightarrow \sum_{n=1}^{\infty} (-1)^n$ diverges since $\lim_{n\to\infty} (-1)^n \neq 0$

$x = 3 \rightarrow \sum_{n=1}^{\infty} (1)^n$ diverges since $\lim_{n\to\infty} 1^n \neq 0$

Interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is $x = (-3, 3)$.

b.) $\sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n 3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3} \right| < 1$$

$$\rightarrow -3 < x < 3$$

$x = -3 \rightarrow \sum_{n=1}^{\infty} \frac{(-3)^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges since $\lim_{n\to\infty} \frac{1}{n} = 0$

$x = 3 \rightarrow \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges p-series $(p=1)$

Interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is $x = [-3, 3)$

3c.) $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{3}\right| < 1$

again... $-3 < x < 3$

$x=-3 \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges since $\lim_{n\to\infty} \frac{1}{n^2} = 0$

$x=3 \rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since p-series $(p=2>1)$

Interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$ is $x=[-3,3]$.

7. $\sum_{n=0}^{\infty} \frac{x^{2n}}{3^n}$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{3^{n+1}} \cdot \frac{3^n}{x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{3}\right| < 1$

$\frac{x^2}{3} < 1 \rightarrow \quad x^2 < 3 \rightarrow -\sqrt{3} < x < \sqrt{3}$

radius of convergence is $\sqrt{3}$

9. $\sum_{n=0}^{\infty} n x^n$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{n x^n}\right| = \lim_{n\to\infty} |x| < 1 \rightarrow -1 < x < 1$

check endpoints

$x=-1 \rightarrow \sum_{n=0}^{\infty} n(-1)^n$ diverges since $\lim_{n\to\infty} n \neq 0$

$x=1 \rightarrow \sum_{n=0}^{\infty} n(1)^n$ diverges since $\lim_{n\to\infty} n \neq 0$

Interval of convergence for $\sum_{n=0}^{\infty} n x^n$ is $x=(-1,1)$.

11. $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n}$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{x^{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{2}\right| < 1$

$x^2 < 2 \rightarrow -\sqrt{2} < x < \sqrt{2}$

check endpoints... (next page)

$$x=\sqrt{2} \rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{(\sqrt{2})^{2n+1}}{2^n n} = \sum_{n=1}^{\infty}(-1)^n \frac{((\sqrt{2})^2)^n \cdot \sqrt{2}}{2^n n} = \sum_{n=1}^{\infty}(-1)^n\left(\frac{\sqrt{2}}{n}\right)$$

$\lim_{n\to\infty} \frac{\sqrt{2}}{n} = 0$ so series converges by AST.

$x=-\sqrt{2} \longrightarrow \sum_{n=1}^{\infty}(-1)^n\left(-\frac{\sqrt{2}}{n}\right)$ again $\lim_{n\to\infty} -\frac{\sqrt{2}}{n} = 0$ so series converges by AST.

Interval of convergence for $\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{2^n n}$ is $x=[-\sqrt{2},\sqrt{2}]$.

13. $\sum_{n=4}^{\infty} \frac{x^n}{n^5}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^5}\cdot\frac{n^5}{x^n}\right| = \lim_{n\to\infty}|x| < 1$$

$-1 < x < 1$, now check endpoints

$x=-1 \rightarrow \sum_{n=4}^{\infty}\frac{(-1)^n}{n^5}$ converges since $\lim_{n\to\infty}\frac{1}{n^5}=0$

$x=1 \rightarrow \sum_{n=4}^{\infty}\frac{1}{n^5}$ converges since p-series $(p=5>1)$.

Interval of convergence for $\sum_{n=4}^{\infty}\frac{x^n}{n^5}$ is $x=[-1,1]$.

15. $\sum_{n=0}^{\infty}\frac{x^n}{(n!)^2}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{((n+1)!)^2}\cdot\frac{(n!)^2}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{(n+1)(n+1)}\right| = 0$$

$\therefore$ Interval of convergence for $\sum_{n=0}^{\infty}\frac{x^n}{(n!)^2}$ is $x=(-\infty,\infty)$.

17. $\sum_{n=0}^{\infty}\frac{(2n)!\,x^n}{(n!)^3}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!\,x^{n+1}}{((n+1)!)^3}\cdot\frac{(n!)^3}{(2n)!\,x^n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\,x}{(n+1)^3}\right|$$

$L=0$ $\therefore$ Interval of convergence for $\sum_{n=0}^{\infty}\frac{(2n)!\,x^n}{(n!)^3}$ is $x=(-\infty,\infty)$.

19. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{(n+1)^2+1}} \cdot \frac{\sqrt{n^2+1}}{x^n} \right| = \lim_{n\to\infty} |x| < 1$

so $-1 < x < 1$, check endpoints

$x = -1 \rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ LCT with divergent $\sum \frac{1}{n}$

$L = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$

$\therefore \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges

$x = 1 \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges since $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$

Interval of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$ is $x = (-1, 1]$

21. $\sum_{n=15}^{\infty} \frac{x^{2n+1}}{3n+1}$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{3n+4} \cdot \frac{3n+1}{x^{2n+1}} \right| = \lim_{n\to\infty} |x^2| < 1$

so $-1 < x < 1$, check endpoints...

$x = -1 \rightarrow \sum_{n=15}^{\infty} \frac{(-1)^{2n+1}}{3n+1} = \sum_{n=15}^{\infty} \frac{-1}{3n+1} = (-1) \sum_{n=15}^{\infty} \frac{1}{3n+1}$

LCT with divergent $\sum \frac{1}{n}$

$L = \lim_{n\to\infty} \frac{\frac{1}{3n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$ $\therefore \sum_{n=15}^{\infty} \frac{1}{3n+1}$ diverges

$x = 1 \rightarrow \sum_{n=15}^{\infty} \frac{1}{3n+1}$ diverges (see above)

Interval of Convergence for $\sum_{n=15}^{\infty} \frac{x^{2n+1}}{3n+1}$ is $x = (-1, 1)$.

23. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n} \right| = \lim_{n\to\infty} |x| < 1 \rightarrow -1 < x < 1$

check endpoints...

$x=-1 \rightarrow \sum_{n=2}^{\infty} \frac{x^n}{\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges since $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

$x=1 \rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln n}$ CT $\rightarrow$ since $\frac{1}{\ln n} \geq \frac{1}{n}$ for $n \geq 2$ and since $\sum \frac{1}{n}$ diverges we know $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ also diverges.

Interval of convergence for $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$ is $x = [-1, 1)$.

25. $\sum_{n=1}^{\infty} n(x-3)^n$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}}{n(x-3)^n} \right| = |x-3| < 1 \rightarrow 2 < x < 4$

$x=2 \rightarrow \sum_{n=1}^{\infty} n(-1)^n$ diverges since $\lim_{n\to\infty} n \neq 0$.

$x=4 \rightarrow \sum_{n=1}^{\infty} n$ diverges since $\lim_{n\to\infty} n \neq 0$.

Interval of convergence for $\sum_{n=1}^{\infty} n(x-3)^n$ is $x = (2, 4)$.

27. $\sum_{n=1}^{\infty} (-1)^n n^5 (x-7)^n$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^5 (x-7)^{n+1}}{n^5 (x-7)^n} \right| = \lim_{n\to\infty} |x-7| < 1 \rightarrow 6 < x < 8$

$x=6 \rightarrow \sum_{n=1}^{\infty} (-1)^n n^5 (-1)^n = \sum_{n=1}^{\infty} n^5$ which diverges $\lim_{n\to\infty} n^5 \neq 0$

$x=8 \rightarrow \sum_{n=1}^{\infty} (-1)^n n^5$ diverges since $\lim_{n\to\infty} n^5 \neq 0$

Interval of convergence for $\sum_{n=1}^{\infty} (-1)^n n^5 (x-7)^n$ is $x = (6, 8)$

29. $\sum_{n=1}^{\infty} \frac{2^n (x+3)^n}{3n}$

$L = \lim_{n\to\infty} \left| \frac{2^{n+1}(x+3)^{n+1}}{3n+3} \cdot \frac{3n}{2^n (x+3)^n} \right| = \lim_{n\to\infty} |2(x+3)| < 1 \rightarrow -\frac{1}{2} < x+3 < \frac{1}{2} \rightarrow -\frac{7}{2} < x < -\frac{5}{2}$

$x = -\frac{7}{2} \rightarrow \sum_{n=1}^{\infty} \frac{2^n (-\frac{1}{2})^n}{3n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$ converges since $\lim_{n\to\infty} \frac{1}{3n} = 0$

$x = -\frac{5}{2} \rightarrow \sum_{n=1}^{\infty} \frac{1}{3n} \rightarrow$ divergent p-series $(p=1)$

Interval of convergence for $\sum_{n=1}^{\infty} \frac{2^n (x+3)^n}{3n}$ is $x = [-\frac{7}{2}, -\frac{5}{2})$.

31. $\sum_{n=0}^{\infty} \frac{(-5)^n (x+10)^n}{n!}$

$L = \lim_{n\to\infty} \left| \frac{5^{n+1}(x+10)^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n (x+10)^n} \right| = \lim_{n\to\infty} \left| \frac{5(x+10)}{n+1} \right| = 0 \quad \therefore$

Interval of convergence for $\sum_{n=0}^{\infty} \frac{(-5)^n (x+10)^n}{n!}$ is $x = (-\infty, \infty)$.

33. $\sum_{n=12}^{\infty} e^n (x-2)^n$

$L = \lim_{n\to\infty} \left| \frac{e^{n+1}(x-2)^{n+1}}{e^n (x-2)^n} \right| = \lim_{n\to\infty} \left| e(x-2) \right| < 1$

$-\frac{1}{e} < x-2 < \frac{1}{e}$

$-\frac{1}{e}+2 < x < \frac{1}{e}+2$

$2-\frac{1}{e} \rightarrow \sum_{n=12}^{\infty} e^n \left(-\frac{1}{e}\right)^n = \sum_{n=12}^{\infty} (-1)^n$ diverges since $\lim_{n\to\infty} (-1)^n \neq 0$

$2+\frac{1}{e} \rightarrow \sum_{n=12}^{\infty} e^n \left(\frac{1}{e}\right)^n = \sum_{n=12}^{\infty} 1^n$ diverges since $\lim_{n\to\infty} 1^n \neq 0$

Interval of convergence for $\sum_{n=12}^{\infty} e^n (x-2)^n$ is $x = (2-\frac{1}{e}, 2+\frac{1}{e})$.