1. $\int \cos^3 x \, dx$ $\qquad \cos^3 x = \cos^2 x \cos x = (1-\sin^2 x)\cos x$

$= \int (1-\sin^2 x)\cos x \, dx$ $\qquad u = \sin x \quad du = \cos x \, dx$

$= \int (1-u^2)\, du = \left[u - \frac{u^3}{3}\right] + C = \sin x - \frac{1}{3}\sin^3 x + C$

3. $\int \sin^3\theta \cos^2\theta \, d\theta$ $\qquad \sin^3\theta = (1-\cos^2\theta)(\sin\theta)$

$= \int (\cos^2\theta - \cos^4\theta)\sin\theta \, d\theta$ $\qquad u = \cos\theta \quad du = -\sin\theta \, d\theta$

$= -\int (u^2 - u^4)\, du = -\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + C = \frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} + C$

5. $\int \sin^3 t \cos^3 t \, dt$ $\qquad \cos^3 t = (1-\sin^2 t)\cos t$

$= \int (\sin^3 t - \sin^5 t)\cos t \, dt$ $\qquad u = \sin t \quad du = \cos t \, dt$

$= \int (u^3 - u^5)\, du = \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\sin^4 t}{4} - \frac{\sin^6 t}{6} + C$

9. $\int \cos^4 y \, dy$ $\qquad \cos^4 y = \left(\frac{1}{2}(1+\cos 2y)\right)^2 = \frac{1}{4}(1 + 2\cos 2y + \cos^2 2y)$

$= \frac{1}{4} + \frac{1}{2}\cos 2y + \frac{1}{8}(1+\cos 4y)$

$= \frac{3}{8} + \frac{1}{2}\cos 2y + \frac{1}{8}\cos 4y$

$= \int \cos^4 y \, dy = \int \left(\frac{3}{8} + \frac{1}{2}\cos 2y + \frac{1}{8}\cos 4y\right) dy$

$= \frac{3}{8}y + \frac{1}{4}\sin 2y + \frac{1}{32}\sin 4y + C$

7. From #1 $\int \cos^3 x \, dx = \sin x - \frac{1}{3}\sin^3 x + C$

$A = \int_0^{\frac{\pi}{2}} \cos^3 x \, dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3 x \, dx = \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}} - \left[\sin x - \frac{1}{3}\sin^3 x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$A = \left(1 - \frac{1}{3}\right) - \left[\left(-1 + \frac{1}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = 3 - 3\left(\frac{1}{3}\right) = 3 - 1 = 2$

11. $\int \sin^4 x \cos^2 x \, dx = \int (\sin^4 x - \sin^6 x)\, dx$

$\cos^2 x = 1 - \sin^2 x$

Use: $\int \sin^n x \, dx = \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} \int \sin^{n-2} x \, dx$

$$\int \sin^4 x = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} \int \sin^2 x \, dx$$

$$\int \sin^6 x \, dx = \frac{-\sin^5 x \cos x}{6} + \frac{5}{6} \int \sin^4 x \, dx$$

so...

$$\int (\sin^4 x - \sin^6 x)\, dx = \frac{\sin^5 x \cos x}{6} + \frac{1}{6}\left[\frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x \, dx\right]$$

$$= \frac{\sin^5 x \cos x}{6} - \frac{\sin^3 x \cos x}{24} + \frac{1}{8} \int \left(\tfrac{1}{2}(1 - \cos 2x)\right) dx$$

$$= \frac{\sin^5 x \cos x}{6} - \frac{\sin^3 x \cos x}{24} + \frac{1}{16}\left[x - \frac{\sin 2x}{2}\right] + C$$

$$= \frac{\sin^5 x \cos x}{6} - \frac{\sin^3 x \cos x}{24} + \frac{x}{16} - \frac{\sin 2x}{32} + C$$

13. Use $\int \sin^m x \cos^n x \, dx = \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \dfrac{m-1}{m+n} \int \sin^{m-2} x \cos^{n-2} x \, dx$

$$\int \sin^3 x \cos^2 x \, dx = \frac{\sin^4 x \cos x}{5} + \frac{2}{5} \int \sin x \, dx$$

$$= \frac{\sin^4 x \cos x}{5} - \frac{2\cos x}{5} + C$$

15. $\int \tan^3 x \sec x$ $\qquad \tan^3 x = \tan^2 x \cdot \tan x = (\sec^2 x - 1)\tan x$

$$= \int (\sec^3 x - \sec x) \tan x \, dx = \int (\sec^2 x - 1) \sec x \tan x \, dx$$

$u = \sec x \rightarrow du = \sec x \tan x \, dx$

$$= \int (u^2 - 1)\, du = \frac{u^3}{3} - u + C = \frac{\sec^3 x}{3} - \sec x + C$$

17. $\int \tan^2 x \sec^4 x$ $\qquad \sec^4 x = \sec^2 x \cdot \sec^2 x = \sec^2 x(\tan^2 x + 1)$

$= \int (\tan^4 x + \tan^2 x) \sec^2 x \, dx = \int (u^4 + u^2) du = \frac{u^5}{5} + \frac{u^3}{3} + C$

$u = \tan x \rightarrow du = \sec^2 x \, dx$

$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

19. $\int \cot^3 x \, dx$ $\qquad \cot^3 x = \cot^2 x \cot x = (\csc^2 x - 1)\cot x$

substitution NOT helpful since $\frac{d}{dx}(\csc x) = -\csc x \cot x$ unless you get creative! :)

let $u = \csc x$

$du = -\csc x \cot x \, dx$

$\int \cot^3 x \, dx = \int \frac{(\csc^2 x - 1)}{-\csc x}(-\csc x)\cot x \, dx = \int \frac{u^2 - 1}{-u} du$

$= \int \left(\frac{1}{u} - u\right) du = \ln|u| - \frac{u^2}{2} + C = \ln|\csc x| - \frac{1}{2}\csc^2 x + C$

21. $\int \cot^5 x \csc^2 x \, dx$ $\qquad \cot^5 x = (\cot^2 x)^2 \cot x = (\csc^2 x - 1)^2 \cot x$

$= (\csc^4 x - 2\csc^2 x + 1)\cot x$

$u = \csc x$

$du = -\csc x \cot x \, dx$

$= (-1)\int (\csc^5 x - 2\csc^3 x + \csc x) \underbrace{\csc x \cot x \, dx \, (-1)}_{du}$

$= -\int (u^5 - 2u^3 + u) du = -\left[\frac{u^6}{6} - \frac{u^4}{2} + \frac{u^2}{2}\right] + C$

$= -\frac{1}{6}\csc^6 x + \frac{1}{2}\csc^4 x + \frac{1}{2}\csc^2 x + C$

23. $\int \cos^5 x \sin x \, dx$ $\qquad u = \cos x \rightarrow du = -\sin x \, dx$

$= -\int u^5 du = -\frac{u^6}{6} + C = -\frac{1}{6}\cos^6 x + C$

25. $\int \cos^4(3x+2)\,dx$

$$\cos^4(3x+2) = \left[\tfrac{1}{2}\left(1+\cos(6x+4)\right)\right]^2 = \tfrac{1}{4}\left(1+2\cos(6x+4)+\cos^2(6x+4)\right)$$
$$= \tfrac{1}{4} + \tfrac{1}{2}\cos(6x+4) + \tfrac{1}{8}\left(1+\cos(12x+8)\right)$$
$$= \tfrac{3}{8} + \tfrac{1}{2}\cos(6x+4) + \tfrac{1}{8}\cos(12x+8)$$
$$= \int\left(\tfrac{3}{8}+\tfrac{1}{2}\cos(6x+4)+\tfrac{1}{8}\cos(12x+8)\right)dx = \tfrac{3}{8}x + \tfrac{1}{12}\sin(6x+4) + \tfrac{1}{96}\sin(12x+8) + C$$

27. $\int \cos^3(\pi\theta)\sin^4(\pi\theta)\,d\theta$ $\qquad \cos^3(\pi\theta) = \cos(\pi\theta)\left[1-\sin^2(\pi\theta)\right]$

$$= \int\left(\sin^4(\pi\theta)-\sin^6(\pi\theta)\right)\cos(\pi\theta)\,d\theta$$

$u = \sin(\pi\theta)$

$du = \cos(\pi\theta)\,d\theta\,(\pi)$

$$= \tfrac{1}{\pi}\int(u^4-u^6)\,du = \tfrac{1}{\pi}\left[\tfrac{u^5}{5}-\tfrac{u^7}{7}\right]+C$$
$$= \tfrac{1}{5\pi}\sin^5(\pi\theta) - \tfrac{1}{7\pi}\sin^7(\pi\theta) + C$$

29. $\int \sin^4(3x)\,dx$

$$\sin^4(3x) = \left[\tfrac{1}{2}(1-\cos 6x)\right]^2 = \tfrac{1}{4}\left(1-2\cos 6x+\cos^2 6x\right)$$
$$= \tfrac{1}{4} - \tfrac{1}{2}\cos 6x + \tfrac{1}{8}(1+\cos 12x) = \tfrac{3}{8} - \tfrac{1}{2}\cos 6x + \tfrac{1}{8}\cos 12x$$
$$= \int\left(\tfrac{3}{8}-\tfrac{1}{2}\cos 6x+\tfrac{1}{8}\cos 12x\right)dx = \tfrac{3}{8}x - \tfrac{1}{12}\sin 6x + \tfrac{1}{96}\sin 12x + C$$

31. $\int \csc^2(3-2x)\,dx = \dfrac{-\cot(3-2x)}{-2} + C = \tfrac{1}{2}\cot(3-2x) + C$

an easy one!

33. $\int \tan x \sec^2 x\, dx$ $\quad u = \tan x \rightarrow du = \sec^2 x\, dx$

$= \int u\, du = \frac{u^2}{2} + C = \frac{1}{2}\tan^2 x + C$

35. $\int \tan^5 x \sec^4 x\, dx$ $\quad \sec^4 x = \sec^2 x\,(\tan^2 x + 1)$

$= \int (\tan^7 x + \tan^5 x)\sec^2 x\, dx$ $\quad u = \tan x \rightarrow du = \sec^2 x\, dx$

$= \int (u^7 + u^5)\, du = \frac{u^8}{8} + \frac{u^6}{6} + C = \frac{1}{8}\tan^8 x + \frac{1}{6}\tan^6 x + C$

37. $\int \tan^6 x \sec^4 x\, dx$ $\quad \sec^4 x = \sec^2 x\,(\tan^2 x + 1)$

$= \int (\tan^8 x + \tan^6 x)\sec^2 x\, dx$ $\quad u = \tan x \rightarrow du = \sec^2 x\, dx$

$= \int (u^8 + u^6)\, du = \frac{u^9}{9} + \frac{u^7}{7} + C = \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C$

39. $\int \cot^5 x \csc^5 x\, dx$ $\quad \cot^5 x = (\csc^2 x - 1)^2 \cot x$

$\quad = (\csc^4 x - 2\csc^2 x + 1)\cot x$

$= \int (\csc^8 x - 2\csc^6 x + \csc^4 x)\csc x \cot x\, dx$ $\quad u = \csc x$, $du = -\csc x \cot x\, dx$

$= -\int (u^8 - 2u^6 + u^4)\, du = -\left[\frac{u^9}{9} - \frac{2u^7}{7} + \frac{u^5}{5}\right] + C$

$= -\frac{1}{9}\csc^9 x + \frac{2}{7}\csc^7 x - \frac{1}{5}\csc^5 x + C$

41. $\int \sin 2x \cos 2x\, dx$ $\quad u = \sin 2x \rightarrow du = 2\cos 2x\, dx$

$= \frac{1}{2}\int u\, du = \frac{1}{4}u^2 + C = \frac{1}{4}\sin^2 2x + C$

43. $\int t\cos^3(t^2)\, dt = \int t\,(1 - \sin^2(t^2))\cos(t^2)\, dt$ $\quad u = \sin(t^2)$, $du = \cos(t^2)2t\, dt$

$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left[u - \frac{u^3}{3}\right] + C = \frac{1}{2}\sin(t^2) - \frac{1}{6}\sin^3(t^2) + C$

45. $\int \cos^2(\sin t)\cos t\,dt$ $\qquad u=\sin t \rightarrow du=\cos t\,dt$

$= \int \cos^2 u\,du = \frac{1}{2}\int(1+\cos 2u)\,du = \frac{1}{2}\left[u+\frac{1}{2}\sin 2u\right]+C$

$= \frac{1}{2}\sin t + \frac{1}{4}\sin(2\sin t) + C$

47. $\int_0^{2\pi} \sin^2 x\,dx = \frac{1}{2}\int_0^{2\pi}(1-\cos 2x)\,dx = \frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{2\pi}$

$= \frac{1}{2}[2\pi] = \pi$

49. $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$ $\qquad \sin^5 x = (\sin^2 x)^2(\sin x) = (1-\cos^2 x)^2(\sin x)$

$= (1-2\cos^2 x+\cos^4 x)(\sin x)$

$= \int_0^{\frac{\pi}{2}}(1-2\cos^2 x+\cos^4 x)(\sin x)\,dx$ $\qquad u=\cos x \quad u(0)=1$, $du=-\sin x\,dx \quad u(\frac{\pi}{2})=0$

$= \int_0^1 (1-2u^2+u^4)\,du = u-\frac{2u^3}{3}+\frac{u^5}{5}\Big|_0^1 = 1-\frac{2}{3}+\frac{1}{5} = \frac{8}{15}$

51. $\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{4}} \sec x\,dx \xrightarrow{\text{Table \#13}} \left[\ln|\sec x+\tan x|\right]_0^{\frac{\pi}{4}}$

$= \ln|\sqrt{2}+1| - \ln|1| = \ln(\sqrt{2}+1)$

53. $\int_0^{\frac{\pi}{3}} \tan x\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx$ $\qquad u=\cos x \quad u(0)=1$, $du=-\sin x\,dx \quad u(\frac{\pi}{3})=\frac{1}{2}$

$= \int_{\frac{1}{2}}^1 \frac{1}{u}\,du = \left[\ln|u|\right]_{\frac{1}{2}}^1 = \ln 1 - \ln(\frac{1}{2}) = -\ln(\frac{1}{2})$

$= \ln 2$

55. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^4 x\,dx$ $\qquad \sec^4 x = (\tan^2 x+1)(\sec^2 x)$

$u=\tan x \rightarrow du=\sec^2 x\,dx \quad u(\frac{\pi}{4})=1,\ u(-\frac{\pi}{4})=-1$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\tan^2 x+1)\sec^2 x\,dx = \int_{-1}^1 (u^2+1)\,du = \left[\frac{u^3}{3}+u\right]_{-1}^1$

$= \frac{4}{3} - \left[-\frac{4}{3}\right] = \frac{8}{3}$

57. $\int_0^{\pi} \sin 3x \cos 4x \, dx$

Must use $\int \sin mx \cos nx \, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C$

$$= \left[-\frac{\cos(-x)}{-2} - \frac{\cos(7x)}{14}\right]_0^{\pi} = \left[\frac{\cos x}{2} - \frac{\cos(7x)}{14}\right]_0^{\pi}$$

$$= \left[\left(\frac{-1}{2} - \frac{-1}{14}\right) - \left(\frac{1}{2} - \frac{1}{14}\right)\right] = -1 + \frac{1}{7} = \frac{-6}{7}$$

59. $\int_0^{\frac{\pi}{6}} \sin 2x \cos 4x \, dx$ same formula as #57

$$= \left[-\frac{\cos(-2x)}{-4} - \frac{\cos(6x)}{12}\right]_0^{\frac{\pi}{6}} = \left[\frac{\cos(2x)}{4} - \frac{\cos(6x)}{12}\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{8} - \frac{-1}{12}\right) - \left(\frac{1}{4} - \frac{1}{12}\right) = \frac{-1}{8} + \frac{1}{6} = \frac{1}{24}$$

61. A more complicated problem was done in #29. See #29.