8.4 HW

1. $f(x) = \sin x$, $a = 0$ $\rightarrow f(0) = 0$

$f'(x) = \cos x$ $\rightarrow f'(0) = 1$

$f''(x) = -\sin x$ $\rightarrow f''(0) = 0$

$f'''(x) = -\cos x$ $\rightarrow f'''(0) = -1$

So the Taylor polynomial approx for $f(x) = \sin x$ centered at $x = 0$ is...

$T_2(x) = f(0) + f'(0)(x-0) + f''(0)/2!\,(x-0)^2$

$T_2(x) = 0 + 1x + 0 = x$

$T_3(x) = T_2(x) + \frac{f'''(0)}{3!}(x-0)^3 = x + \frac{-1}{6}x^3 = x - \frac{1}{6}x^3$

3. $f(x) = \frac{1}{1+x}$, $a = 2$ $\rightarrow f(2) = \frac{1}{3}$

$f'(x) = -1(1+x)^{-2} = \frac{-1}{(1+x)^2}$ $\rightarrow f'(2) = \frac{-1}{9}$

$f''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$ $\rightarrow f''(2) = \frac{2}{27}$

$f'''(x) = -6(1+x)^{-4} = \frac{-6}{(1+x)^4}$ $\rightarrow f'''(3) = \frac{-2}{27}$

The Taylor poly approx for $f(x) = \frac{1}{1+x}$ centered at $x = 2$ is

$T_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$

$T_2(x) = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2$

$T_3(x) = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3$

5. $f(x) = x^4 - 2x$, $a = 3$ $\rightarrow f(3) = 81 - 6 = 75$

$f'(x) = 4x^3 - 2$ $\rightarrow f'(3) = 106$

$f''(x) = 12x^2$ $\rightarrow f''(3) = 108$

$f'''(x) = 24x$ $\rightarrow f'''(3) = 72$

Taylor poly approx for $f(x) = x^4 - 2x$ at $x = 3$ is...

$T_2(x) = f(3) + f'(3)(x-3) + f''(3)/2!\,(x-3)^2$

$T_2(x) = 75 + 106(x-3) + 54(x-3)^2$

$T_3(x) = 75 + 106(x-3) + 54(x-3)^2 + 12(x-3)^3$

7. $f(x) = \tan x$, $a = 0$ $\rightarrow$ $f(0) = 0$

$f'(x) = \sec^2 x$ $\rightarrow$ $f'(0) = \left(\frac{1}{\cos(0)}\right)^2 = 1$

$f''(x) = 2\sec^2 x \tan x$ $\rightarrow$ $f''(0) = 0$

$f'''(x) = 4\sec x \tan x + 2\sec^4 x$ $\rightarrow$ $f'''(0) = 4(1)(0) + 2(1) = 2$

$T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$

$T_2(x) = 0 + 1x + 0 = x$

$T_3(x) = x + \frac{2}{3!}(x-0)^3 = x + \frac{1}{3}x^3$

9. $f(x) = e^{-x} + e^{-2x}$, $a = 0$ $\rightarrow$ $f(0) = e^0 + e^0 = 2$

$f'(x) = -e^{-x} - 2e^{-2x}$ $\rightarrow$ $f'(0) = -1 - 2 = -3$

$f''(x) = e^{-x} + 4e^{-2x}$ $\rightarrow$ $f''(0) = 1 + 4 = 5$

$f'''(x) = -e^{-x} - 8e^{-2x}$ $\rightarrow$ $f'''(0) = -1 - 8 = -9$

$T_2(x) = f(0) + f'(0)(x-0) + f''(0)/2!\,(x-0)^2$

$T_2(x) = 2 - 3x + \frac{5}{2}x^2$

$T_3(x) = 2 - 3x + \frac{5}{2}x^2 - \frac{3}{2}x^3$

11. $f(x) = x^2 e^{-x}$, $a = 1$ $\rightarrow$ $f(1) = 1e^{-1} = \frac{1}{e}$

$f'(x) = 2xe^{-x} + x^2(-e^{-x}) = e^{-x}(2x - x^2)$ $\rightarrow$ $f'(1) = \frac{1}{e}$

$f''(x) = -e^{-x}(2x-x^2) + e^{-x}(2-2x) = e^{-x}(x^2-4x+2) \rightarrow f''(1) = \frac{-1}{e}$

$f'''(x) = -e^{-x}(x^2-4x+2) + e^{-x}(2x-4) = e^{-x}(-x^2+6x-6) \rightarrow f'''(1) = \frac{-1}{e}$

$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$

$T_2(x) = \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{2e}(x-1)^2$

$T_3(x) = \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{2e}(x-1)^2 - \frac{1}{6e}(x-1)^3$

13. $f(x) = \frac{\ln x}{x}$ $a = 1$ $\rightarrow$ $f(1) = \frac{\ln 1}{1} = 0$

$f'(x) = \frac{x(\frac{1}{x}) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$ $\rightarrow$ $f'(1) = \frac{1}{1^2} = 1$

$f''(x) = \frac{x^2(\frac{-1}{x}) - (1-\ln x)(2x)}{x^4} = \frac{-x - 2x + 2x\ln x}{x^4} = \frac{-3 + 2\ln x}{x^3} \rightarrow f''(1) = -3$

$f'''(x) = \frac{x^3(\frac{2}{x}) - (-3+2\ln x)(3x^2)}{x^6} = \frac{2x^2 + 9x^2 - 6x^2\ln x}{x^6} = \frac{11 - 6\ln x}{x^4} \rightarrow f'''(1) = 11$

$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)$

$T_2(x) = 0 + 1(x-1) - \frac{3}{2}(x-1)^2 = x - 1 - \frac{3}{2}(x-1)^2$

$T_3(x) = x - 1 - \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3$

15. $f(x) = e^x$, $a = 0$

$f'(x) = f''(x) = f'''(x) \ldots = f^{(n)}(x) = e^x$ $\rightarrow$ $f^{(n)}(0) = e^0 = 1$

So...

$T_n(x) = 1 + 1(x-0)^1 + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^2 + \cdots + \frac{1}{n!}(x-0)^n$

$T_n(x) = 1 + 1x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \frac{1}{n!}x^n$

17. $f(x) = \sin x$, $a = 0$ $\rightarrow f(0) = 0$

$f'(x) = \cos x \rightarrow f'(0) = 1$

$f''(x) = -\sin x \rightarrow f''(0) = 0$

$f'''(x) = -\cos x \rightarrow f'''(0) = -1$

$$T_3(x) = 0 + \frac{1}{1!}x^1 + 0 + \frac{(-1)}{3!}x^3 + \cdots$$

Notice all even terms $(0, 2, 4 \ldots)$ will result in a zero

Hence, $T_n(x)$ can be represented by $T_{2n+1}(x)$. All

the $(2n+1)^{st}$ term will have a value (since $\cos(0) = 1$).

$$T_{(2n+1)}(x) = \frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{(2n+1)}$$

for $n \geq 0$.

Since the $(2n+2)$ term will result in a zero $(\sin(0) = 0)$,

we know $T_{(2n+1)}(x) = T_{(2n+2)}(x)$