This is true because $a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2$. Now, since $a_1 < 2$, we can apply (3) to conclude that $a_2 < 2$. Similarly, $a_2 < 2$ implies $a_3 < 2$, and so on for all n. Formally speaking, this is a proof by induction.

Step 2. **Show that $\{a_n\}$ is increasing.**
Since a_n is positive and $a_n < 2$, we have

$$a_{n+1} = \sqrt{2a_n} > \sqrt{a_n \cdot a_n} = a_n$$

This shows that $\{a_n\}$ is increasing. ■

We conclude that the limit L exists and hence $L = 2$.

10.1 SUMMARY

- A sequence $\{a_n\}$ *converges* to a limit L if, for every $\epsilon > 0$, there is a number M such that

$$|a_n - L| < \epsilon \qquad \text{for all } n > M$$

We write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.
- If no limit exists, we say that $\{a_n\}$ *diverges*.
- In particular, if the terms increase without bound, we say that $\{a_n\}$ diverges to infinity.
- If $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.
- A *geometric sequence* is a sequence $a_n = cr^n$, where c and r are nonzero.
- The Basic Limit Laws and the Squeeze Theorem apply to sequences.
- If $f(x)$ is continuous and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.
- A sequence $\{a_n\}$ is
  - *bounded above* by M if $a_n \le M$ for all n.
  - *bounded below* by m if $a_n \ge m$ for all n.

If $\{a_n\}$ is bounded above and below, $\{a_n\}$ is called *bounded*.
- A sequence $\{a_n\}$ is *monotonic* if it is increasing ($a_n < a_{n+1}$) or decreasing ($a_n > a_{n+1}$).
- Bounded monotonic sequences converge (Theorem 6).

10.1 EXERCISES

Preliminary Questions

1. What is a_4 for the sequence $a_n = n^2 - n$?

2. Which of the following sequences converge to zero?

(a) $\dfrac{n^2}{n^2+1}$ **(b)** 2^n **(c)** $\left(\dfrac{-1}{2}\right)^n$

3. Let a_n be the nth decimal approximation to $\sqrt{2}$. That is, $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, etc. What is $\lim_{n\to\infty} a_n$?

4. Which of the following sequences is defined recursively?

(a) $a_n = \sqrt{4+n}$ **(b)** $b_n = \sqrt{4+b_{n-1}}$

5. Theorem 5 says that every convergent sequence is bounded. Determine if the following statements are true or false and if false, give a counterexample.

(a) If $\{a_n\}$ is bounded, then it converges.

(b) If $\{a_n\}$ is not bounded, then it diverges.

(c) If $\{a_n\}$ diverges, then it is not bounded.

Exercises

1. Match each sequence with its general term:

$a_1, a_2, a_3, a_4, \ldots$	General term
(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$	(i) $\cos \pi n$
(b) $-1, 1, -1, 1, \ldots$	(ii) $\frac{n!}{2^n}$
(c) $1, -1, 1, -1, \ldots$	(iii) $(-1)^{n+1}$
(d) $\frac{1}{2}, \frac{2}{4}, \frac{6}{8}, \frac{24}{16} \ldots$	(iv) $\frac{n}{n+1}$

2. Let $a_n = \dfrac{1}{2n-1}$ for $n = 1, 2, 3, \ldots$. Write out the first three terms of the following sequences.

(a) $b_n = a_{n+1}$ **(b)** $c_n = a_{n+3}$

(c) $d_n = a_n^2$ **(d)** $e_n = 2a_n - a_{n+1}$

In Exercises 3–12, calculate the first four terms of the sequence, starting with $n = 1$.

3. $c_n = \dfrac{3^n}{n!}$ **4.** $b_n = \dfrac{(2n-1)!}{n!}$

5. $a_1 = 2, \quad a_{n+1} = 2a_n^2 - 3$

6. $b_1 = 1, \quad b_n = b_{n-1} + \dfrac{1}{b_{n-1}}$

7. $b_n = 5 + \cos \pi n$ **8.** $c_n = (-1)^{2n+1}$

9. $c_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$

10. $a_n = n + (n+1) + (n+2) + \cdots + (2n)$

11. $b_1 = 2, \quad b_2 = 3, \quad b_n = 2b_{n-1} + b_{n-2}$

12. $c_n = n$-place decimal approximation to e

13. Find a formula for the nth term of each sequence.

(a) $\dfrac{1}{1}, \dfrac{-1}{8}, \dfrac{1}{27}, \ldots$ **(b)** $\dfrac{2}{6}, \dfrac{3}{7}, \dfrac{4}{8}, \ldots$

14. Suppose that $\lim_{n\to\infty} a_n = 4$ and $\lim_{n\to\infty} b_n = 7$. Determine:

(a) $\lim_{n\to\infty} (a_n + b_n)$ **(b)** $\lim_{n\to\infty} a_n^3$

(c) $\lim_{n\to\infty} \cos(\pi b_n)$ **(d)** $\lim_{n\to\infty} (a_n^2 - 2a_n b_n)$

In Exercises 15–26, use Theorem 1 to determine the limit of the sequence or state that the sequence diverges.

15. $a_n = 12$ **16.** $a_n = 20 - \dfrac{4}{n^2}$

17. $b_n = \dfrac{5n-1}{12n+9}$ **18.** $a_n = \dfrac{4+n-3n^2}{4n^2+1}$

19. $c_n = -2^{-n}$ **20.** $z_n = \left(\dfrac{1}{3}\right)^n$

21. $c_n = 9^n$ **22.** $z_n = 10^{-1/n}$

23. $a_n = \dfrac{n}{\sqrt{n^2+1}}$ **24.** $a_n = \dfrac{n}{\sqrt{n^3+1}}$

25. $a_n = \ln\left(\dfrac{12n+2}{-9+4n}\right)$ **26.** $r_n = \ln n - \ln(n^2+1)$

In Exercises 27–30, use Theorem 4 to determine the limit of the sequence.

27. $a_n = \sqrt{4 + \dfrac{1}{n}}$ **28.** $a_n = e^{4n/(3n+9)}$

29. $a_n = \cos^{-1}\left(\dfrac{n^3}{2n^3+1}\right)$ **30.** $a_n = \tan^{-1}(e^{-n})$

31. Let $a_n = \dfrac{n}{n+1}$. Find a number M such that:

(a) $|a_n - 1| \le 0.001$ for $n \ge M$.

(b) $|a_n - 1| \le 0.00001$ for $n \ge M$.

Then use the limit definition to prove that $\lim_{n\to\infty} a_n = 1$.

32. Let $b_n = \left(\frac{1}{3}\right)^n$.

(a) Find a value of M such that $|b_n| \le 10^{-5}$ for $n \ge M$.

(b) Use the limit definition to prove that $\lim_{n\to\infty} b_n = 0$.

33. Use the limit definition to prove that $\lim_{n\to\infty} n^{-2} = 0$.

34. Use the limit definition to prove that $\lim_{n\to\infty} \dfrac{n}{n+n^{-1}} = 1$.

In Exercises 35–62, use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges.

35. $a_n = 10 + \left(-\dfrac{1}{9}\right)^n$ **36.** $d_n = \sqrt{n+3} - \sqrt{n}$

37. $c_n = 1.01^n$ **38.** $b_n = e^{1-n^2}$

39. $a_n = 2^{1/n}$ **40.** $b_n = n^{1/n}$

41. $c_n = \dfrac{9^n}{n!}$ **42.** $a_n = \dfrac{8^{2n}}{n!}$

43. $a_n = \dfrac{3n^2+n+2}{2n^2-3}$ **44.** $a_n = \dfrac{\sqrt{n}}{\sqrt{n}+4}$

45. $a_n = \dfrac{\cos n}{n}$ **46.** $c_n = \dfrac{(-1)^n}{\sqrt{n}}$

47. $d_n = \ln 5^n - \ln n!$

48. $d_n = \ln(n^2+4) - \ln(n^2-1)$

49. $a_n = \left(2 + \dfrac{4}{n^2}\right)^{1/3}$ **50.** $b_n = \tan^{-1}\left(1 - \dfrac{2}{n}\right)$

51. $c_n = \ln\left(\dfrac{2n+1}{3n+4}\right)$ **52.** $c_n = \dfrac{n}{n+n^{1/n}}$

53. $y_n = \dfrac{e^n}{2^n}$ **54.** $a_n = \dfrac{n}{2^n}$

55. $y_n = \dfrac{e^n + (-3)^n}{5^n}$

56. $b_n = \dfrac{(-1)^n n^3 + 2^{-n}}{3n^3 + 4^{-n}}$

57. $a_n = n \sin \dfrac{\pi}{n}$

58. $b_n = \dfrac{n!}{\pi^n}$

59. $b_n = \dfrac{3 - 4^n}{2 + 7 \cdot 4^n}$

60. $a_n = \dfrac{3 - 4^n}{2 + 7 \cdot 3^n}$

61. $a_n = \left(1 + \dfrac{1}{n}\right)^n$

62. $a_n = \left(1 + \dfrac{1}{n^2}\right)^n$

In Exercises 63–66, find the limit of the sequence using L'Hôpital's Rule.

63. $a_n = \dfrac{(\ln n)^2}{n}$

64. $b_n = \sqrt{n} \ln \left(1 + \dfrac{1}{n}\right)$

65. $c_n = n\left(\sqrt{n^2 + 1} - n\right)$

66. $d_n = n^2\left(\sqrt[3]{n^3 + 1} - n\right)$

In Exercises 67–70, use the Squeeze Theorem to evaluate $\lim\limits_{n\to\infty} a_n$ *by verifying the given inequality.*

67. $a_n = \dfrac{1}{\sqrt{n^4 + n^8}}, \quad \dfrac{1}{\sqrt{2n^4}} \le a_n \le \dfrac{1}{\sqrt{2n^2}}$

68. $c_n = \dfrac{1}{\sqrt{n^2 + 1}} + \dfrac{1}{\sqrt{n^2 + 2}} + \cdots + \dfrac{1}{\sqrt{n^2 + n}},$

$\dfrac{n}{\sqrt{n^2 + n}} \le c_n \le \dfrac{n}{\sqrt{n^2 + 1}}$

69. $a_n = (2^n + 3^n)^{1/n}, \quad 3 \le a_n \le (2 \cdot 3^n)^{1/n} = 2^{1/n} \cdot 3$

70. $a_n = (n + 10^n)^{1/n}, \quad 10 \le a_n \le (2 \cdot 10^n)^{1/n}$

71. Which of the following statements is equivalent to the assertion $\lim\limits_{n\to\infty} a_n = L$? Explain.

(a) For every $\epsilon > 0$, the interval $(L - \epsilon, L + \epsilon)$ contains at least one element of the sequence $\{a_n\}$.

(b) For every $\epsilon > 0$, the interval $(L - \epsilon, L + \epsilon)$ contains all but at most finitely many elements of the sequence $\{a_n\}$.

72. Show that $a_n = \dfrac{1}{2n + 1}$ is decreasing.

73. Show that $a_n = \dfrac{3n^2}{n^2 + 2}$ is increasing. Find an upper bound.

74. Show that $a_n = \sqrt[3]{n + 1} - n$ is decreasing.

75. Give an example of a divergent sequence $\{a_n\}$ such that $\lim\limits_{n\to\infty} |a_n|$ converges.

76. Give an example of *divergent* sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n + b_n\}$ converges.

77. Using the limit definition, prove that if $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

78. Use the limit definition to prove that if $\{a_n\}$ is a convergent sequence of integers with limit L, then there exists a number M such that $a_n = L$ for all $n \ge M$.

79. Theorem 1 states that if $\lim\limits_{x\to\infty} f(x) = L$, then the sequence $a_n = f(n)$ converges and $\lim\limits_{n\to\infty} a_n = L$. Show that the *converse* is false. In other words, find a function $f(x)$ such that $a_n = f(n)$ converges but $\lim\limits_{x\to\infty} f(x)$ does not exist.

80. Use the limit definition to prove that the limit does not change if a finite number of terms are added or removed from a convergent sequence.

81. Let $b_n = a_{n+1}$. Use the limit definition to prove that if $\{a_n\}$ converges, then $\{b_n\}$ also converges and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

82. Let $\{a_n\}$ be a sequence such that $\lim\limits_{n\to\infty} |a_n|$ exists and is nonzero. Show that $\lim\limits_{n\to\infty} a_n$ exists if and only if there exists an integer M such that the sign of a_n does not change for $n > M$.

83. Proceed as in Example 12 to show that the sequence $\sqrt{3}$, $\sqrt{3\sqrt{3}}$, $\sqrt{3\sqrt{3\sqrt{3}}}$, ... is increasing and bounded above by $M = 3$. Then prove that the limit exists and find its value.

84. Let $\{a_n\}$ be the sequence defined recursively by

$$a_0 = 0, \qquad a_{n+1} = \sqrt{2 + a_n}$$

Thus, $a_1 = \sqrt{2}$, $\quad a_2 = \sqrt{2 + \sqrt{2}}$, $\quad a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \dots$.

(a) Show that if $a_n < 2$, then $a_{n+1} < 2$. Conclude by induction that $a_n < 2$ for all n.

(b) Show that if $a_n < 2$, then $a_n \le a_{n+1}$. Conclude by induction that $\{a_n\}$ is increasing.

(c) Use (a) and (b) to conclude that $L = \lim\limits_{n\to\infty} a_n$ exists. Then compute L by showing that $L = \sqrt{2 + L}$.

Further Insights and Challenges

85. Show that $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \infty$. *Hint:* Verify that $n! \ge (n/2)^{n/2}$ by observing that half of the factors of $n!$ are greater than or equal to $n/2$.

86. Let $b_n = \dfrac{\sqrt[n]{n!}}{n}$.

(a) Show that $\ln b_n = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \ln \dfrac{k}{n}$.

(b) Show that $\ln b_n$ converges to $\displaystyle\int_0^1 \ln x\, dx$, and conclude that $b_n \to e^{-1}$.