**CONCEPTUAL INSIGHT** The convergence of an infinite series $\sum a_n$ depends on two factors: (1) how quickly $a_n$ tends to zero, and (2) how much cancellation takes place among the terms. Consider

Harmonic series (diverges): $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

$p$-Series with $p = 2$ (converges): $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$

Alternating harmonic series (converges): $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

The harmonic series diverges because reciprocals $1/n$ do not tend to zero quickly enough. By contrast, the reciprocal squares $1/n^2$ tend to zero quickly enough for the $p$-series with $p = 2$ to converge. The alternating harmonic series converges, but only due to the cancellation among the terms.

## 10.4 SUMMARY

- $\sum a_n$ *converges absolutely* if the positive series $\sum |a_n|$ converges.
- Absolute convergence implies convergence: If $\sum |a_n|$ converges, then $\sum a_n$ also converges.
- $\sum a_n$ *converges conditionally* if $\sum a_n$ converges but $\sum |a_n|$ diverges.
- *Leibniz Test:* If $\{a_n\}$ is positive and decreasing and $\lim_{n\to\infty} a_n = 0$, then the alternating series

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

converges. Furthermore, $|S - S_N| < a_{N+1}$.

- We have developed two ways to handle nonpositive series: Show absolute convergence if possible, or use the Leibniz Test, if applicable.

## 10.4 EXERCISES

### Preliminary Questions

**1.** Give an example of a series such that $\sum a_n$ converges but $\sum |a_n|$ diverges.

**2.** Which of the following statements is equivalent to Theorem 1?

**(a)** If $\sum_{n=0}^{\infty} |a_n|$ diverges, then $\sum_{n=0}^{\infty} a_n$ also diverges.

**(b)** If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} |a_n|$ also diverges.

**(c)** If $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} |a_n|$ also converges.

**3.** Lathika argues that $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ is an alternating series and therefore converges. Is Lathika right?

**4.** Suppose that $a_n$ is positive, decreasing, and tends to 0, and let $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$. What can we say about $|S - S_{100}|$ if $a_{101} = 10^{-3}$? Is $S$ larger or smaller than $S_{100}$?

## Exercises

**1.** Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$$

converges absolutely.

**2.** Show that the following series converges conditionally:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2/3}} = \frac{1}{1^{2/3}} - \frac{1}{2^{2/3}} + \frac{1}{3^{2/3}} - \frac{1}{4^{2/3}} + \cdots$$

*In Exercises 3–10, determine whether the series converges absolutely, conditionally, or not at all.*

**3.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$

**4.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$

**5.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(1.1)^n}$

**6.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi n}{4})}{n^2}$

**7.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

**8.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\frac{1}{n}}$

**9.** $\displaystyle\sum_{n=2}^{\infty} \frac{\cos n\pi}{(\ln n)^2}$

**10.** $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

**11.** Let $S = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$.

**(a)** Calculate $S_n$ for $1 \le n \le 10$.

**(b)** Use Eq. (2) to show that $0.9 \le S \le 0.902$.

**12.** Use Eq. (2) to approximate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$$

to four decimal places.

**13.** Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ to three decimal places.

**14.** CAS Let

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$$

Use a computer algebra system to calculate and plot the partial sums $S_n$ for $1 \le n \le 100$. Observe that the partial sums zigzag above and below the limit.

*In Exercises 15–16, find a value of N such that $S_N$ approximates the series with an error of at most $10^{-5}$. If you have a CAS, compute this value of $S_N$.*

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)(n+3)}$

**16.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n!}$

*In Exercises 17–32, determine convergence or divergence by any method.*

**17.** $\displaystyle\sum_{n=0}^{\infty} 7^{-n}$

**18.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{7.5}}$

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n - 3^n}$

**20.** $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2 - n}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^4 + 12n}$

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

**24.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+1}}$

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{5^n}$

**26.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$

**27.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n^3/3}$

**28.** $\displaystyle\sum_{n=1}^{\infty} n e^{-n^3/3}$

**29.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1/2}(\ln n)^2}$

**30.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/4}}$

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.05}}$

**32.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

**33.** Show that

$$S = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots$$

converges by computing the partial sums. Does it converge absolutely?

**34.** The Leibniz Test cannot be applied to

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \cdots$$

Why not? Show that it converges by another method.

**35.** **Assumptions Matter** Show by counterexample that the Leibniz Test does not remain true if the sequence $a_n$ tends to zero but is not assumed nonincreasing. *Hint:* Consider

$$R = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \cdots + \left(\frac{1}{n} - \frac{1}{2^n}\right) + \cdots$$

**36.** Determine whether the following series converges conditionally:

$$1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{7} + \frac{1}{4} - \frac{1}{9} + \frac{1}{5} - \frac{1}{11} + \cdots$$

**37.** Prove that if $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges. Then give an example where $\sum a_n$ is only conditionally convergent and $\sum a_n^2$ diverges.