10.5 SUMMARY

- *Ratio Test:* Assume that $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists. Then $\sum a_n$
  - Converges absolutely if $\rho < 1$.
  - Diverges if $\rho > 1$.
  - Inconclusive if $\rho = 1$.
- *Root Test:* Assume that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists. Then $\sum a_n$
  - Converges absolutely if $L < 1$.
  - Diverges if $L > 1$.
  - Inconclusive if $L = 1$.

10.5 EXERCISES

Preliminary Questions

1. In the Ratio Test, is ρ equal to $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$?

2. Is the Ratio Test conclusive for $\sum_{n=1}^{\infty} \frac{1}{2^n}$? Is it conclusive for $\sum_{n=1}^{\infty} \frac{1}{n}$?

3. Can the Ratio Test be used to show convergence if the series is only conditionally convergent?

Exercises

In Exercises 1–20, apply the Ratio Test to determine convergence or divergence, or state that the Ratio Test is inconclusive.

1. $\sum_{n=1}^{\infty} \frac{1}{5^n}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{5^n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

4. $\sum_{n=0}^{\infty} \frac{3n+2}{5n^3+1}$

5. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

6. $\sum_{n=1}^{\infty} \frac{2^n}{n}$

7. $\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$

8. $\sum_{n=1}^{\infty} \frac{n^3}{3^{n^2}}$

9. $\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$

10. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

11. $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$

12. $\sum_{n=1}^{\infty} \frac{n^{40}}{n!}$

13. $\sum_{n=0}^{\infty} \frac{n!}{6^n}$

14. $\sum_{n=1}^{\infty} \frac{n!}{n^9}$

15. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

16. $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

17. $\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

18. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

19. $\sum_{n=2}^{\infty} \frac{1}{2^n+1}$

20. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

21. Show that $\sum_{n=1}^{\infty} n^k 3^{-n}$ converges for all exponents k.

22. Show that $\sum_{n=1}^{\infty} n^2 x^n$ converges if $|x| < 1$.

23. Show that $\sum_{n=1}^{\infty} 2^n x^n$ converges if $|x| < \frac{1}{2}$.

24. Show that $\sum_{n=1}^{\infty} \frac{r^n}{n!}$ converges for all r.

25. Show that $\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges if $|r| < 1$.

26. Is there any value of k such that $\sum_{n=1}^{\infty} \frac{2^n}{n^k}$ converges?

27. Show that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. *Hint:* Use $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

In Exercises 28–33, assume that $|a_{n+1}/a_n|$ converges to $\rho = \frac{1}{3}$. What can you say about the convergence of the given series?

28. $\sum_{n=1}^{\infty} n a_n$ **29.** $\sum_{n=1}^{\infty} n^3 a_n$ **30.** $\sum_{n=1}^{\infty} 2^n a_n$

31. $\sum_{n=1}^{\infty} 3^n a_n$ **32.** $\sum_{n=1}^{\infty} 4^n a_n$ **33.** $\sum_{n=1}^{\infty} a_n^2$

34. Assume that $|a_{n+1}/a_n|$ converges to $\rho = 4$. Does $\sum_{n=1}^{\infty} a_n^{-1}$ converge (assume that $a_n \neq 0$ for all n)?

35. Is the Ratio Test conclusive for the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$?

In Exercises 36–41, use the Root Test to determine convergence or divergence (or state that the test is inconclusive).

36. $\sum_{n=0}^{\infty} \frac{1}{10^n}$ **37.** $\sum_{n=1}^{\infty} \frac{1}{n^n}$

38. $\sum_{k=0}^{\infty} \left(\frac{k}{k+10}\right)^k$ **39.** $\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$

40. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n}$ **41.** $\sum_{n=4}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

42. Prove that $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$ diverges. *Hint:* Use $2^{n^2} = (2^n)^n$ and $n! \le n^n$.

In Exercises 43–56, determine convergence or divergence using any method covered in the text so far.

43. $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$ **44.** $\sum_{n=1}^{\infty} \frac{n^3}{n!}$

45. $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ **46.** $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

47. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$ **48.** $\sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$

49. $\sum_{n=1}^{\infty} n^{-0.8}$ **50.** $\sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$

51. $\sum_{n=1}^{\infty} 4^{-2n+1}$ **52.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

53. $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ **54.** $\sum_{n=1}^{\infty} (-1)^n \cos \frac{1}{n}$

55. $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$ **56.** $\sum_{n=1}^{\infty} \left(\frac{n}{n+12}\right)^n$

Further Insights and Challenges

57. **Proof of the Root Test** Let $S = \sum_{n=0}^{\infty} a_n$ be a positive series, and assume that $L = \lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

(a) Show that S converges if $L < 1$. *Hint:* Choose R with $L < R < 1$ and show that $a_n \le R^n$ for n sufficiently large. Then compare with the geometric series $\sum R^n$.

(b) Show that S diverges if $L > 1$.

58. Show that the Ratio Test does not apply, but verify convergence using the Comparison Test for the series

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \cdots$$

59. Let $S = \sum_{n=1}^{\infty} \frac{c^n n!}{n^n}$, where c is a constant.

(a) Prove that S converges absolutely if $|c| < e$ and diverges if $|c| > e$.

(b) It is known that $\lim_{n\to\infty} \frac{e^n n!}{n^{n+1/2}} = \sqrt{2\pi}$. Verify this numerically.

(c) Use the Limit Comparison Test to prove that S diverges for $c = e$.

10.6 Power Series

A **power series** with center c is an infinite series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

where x is a variable. For example,

$$F(x) = 1 + (x-2) + 2(x-2)^2 + 3(x-2)^3 + \cdots \qquad \boxed{1}$$

is a power series with center $c = 2$.