Applying this recursion relation k times, we obtain the closed formula

$$a_{2k+1} = (-1)^k \left(\frac{1}{4k(k+1)}\right)\left(\frac{1}{4(k-1)k}\right)\cdots\left(\frac{1}{4(1)(2)}\right) = \frac{(-1)^k}{4^k\, k!\,(k+1)!}$$

Thus we obtain a power series representation of our solution:

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k!(k+1)!} x^{2k+1}$$

A straightforward application of the Ratio Test shows that $F(x)$ has an infinite radius of convergence. Therefore, $F(x)$ is a solution of the initial value problem for all x. ■

10.6 SUMMARY

- A *power series* is an infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

The constant c is called the *center* of $F(x)$.

- Every power series $F(x)$ has a *radius of convergence* R (Figure 5) such that
  - $F(x)$ converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.
  - $F(x)$ may converge or diverge at the endpoints $c-R$ and $c+R$.

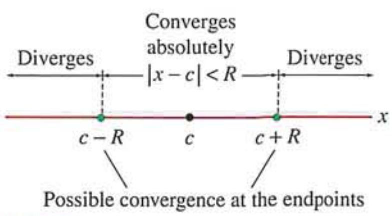

FIGURE 5 Interval of convergence of a power series.

We set $R = 0$ if $F(x)$ converges only for $x = c$ and $R = \infty$ if $F(x)$ converges for all x.

- The *interval of convergence* of $F(x)$ consists of the open interval $(c-R, c+R)$ and possibly one or both endpoints $c-R$ and $c+R$.
- In many cases, the Ratio Test can be used to find the radius of convergence R. It is necessary to check convergence at the endpoints separately.
- If $R > 0$, then $F(x)$ is differentiable on $(c-R, c+R)$ and

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}, \qquad \int F(x)\,dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

(A is any constant). These two power series have the same radius of convergence R.

- The expansion $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is valid for $|x| < 1$. It can be used to derive expansions of other related functions by substitution, integration, or differentiation.

10.6 EXERCISES

Preliminary Questions

1. Suppose that $\sum a_n x^n$ converges for $x = 5$. Must it also converge for $x = 4$? What about $x = -3$?

2. Suppose that $\sum a_n (x-6)^n$ converges for $x = 10$. At which of the points (a)–(d) must it also converge?

(a) $x = 8$ **(b)** $x = 11$ **(c)** $x = 3$ **(d)** $x = 0$

3. What is the radius of convergence of $F(3x)$ if $F(x)$ is a power series with radius of convergence $R = 12$?

4. The power series $F(x) = \sum_{n=1}^{\infty} n x^n$ has radius of convergence $R = 1$. What is the power series expansion of $F'(x)$ and what is its radius of convergence?

Exercises

1. Use the Ratio Test to determine the radius of convergence R of $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$. Does it converge at the endpoints $x = \pm R$?

2. Use the Ratio Test to show that $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}2^n}$ has radius of convergence $R = 2$. Then determine whether it converges at the endpoints $R = \pm 2$.

3. Show that the power series (a)–(c) have the same radius of convergence. Then show that (a) diverges at both endpoints, (b) converges at one endpoint but diverges at the other, and (c) converges at both endpoints.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ **(c)** $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$

4. Repeat Exercise 3 for the following series:

(a) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{9^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n9^n}$ **(c)** $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 9^n}$

5. Show that $\sum_{n=0}^{\infty} n^n x^n$ diverges for all $x \neq 0$.

6. For which values of x does $\sum_{n=0}^{\infty} n! x^n$ converge?

7. Use the Ratio Test to show that $\sum_{n=0}^{\infty} \frac{x^{2n}}{3^n}$ has radius of convergence $R = \sqrt{3}$.

8. Show that $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{64^n}$ has radius of convergence $R = 4$.

In Exercises 9–34, find the interval of convergence.

9. $\sum_{n=0}^{\infty} nx^n$

10. $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$

11. $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n}$

12. $\sum_{n=0}^{\infty} (-1)^n \frac{n}{4^n} x^{2n}$

13. $\sum_{n=4}^{\infty} \frac{x^n}{n^5}$

14. $\sum_{n=8}^{\infty} n^7 x^n$

15. $\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$

16. $\sum_{n=0}^{\infty} \frac{8^n}{n!} x^n$

17. $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^3} x^n$

18. $\sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!} x^{2n-1}$

19. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$

20. $\sum_{n=0}^{\infty} \frac{x^n}{n^4+2}$

21. $\sum_{n=15}^{\infty} \frac{x^{2n+1}}{3n+1}$

22. $\sum_{n=1}^{\infty} \frac{x^n}{n - 4\ln n}$

23. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

24. $\sum_{n=2}^{\infty} \frac{x^{3n+2}}{\ln n}$

25. $\sum_{n=1}^{\infty} n(x-3)^n$

26. $\sum_{n=1}^{\infty} \frac{(-5)^n (x-3)^n}{n^2}$

27. $\sum_{n=1}^{\infty} (-1)^n n^5 (x-7)^n$

28. $\sum_{n=0}^{\infty} 27^n (x-1)^{3n+2}$

29. $\sum_{n=1}^{\infty} \frac{2^n}{3n} (x+3)^n$

30. $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$

31. $\sum_{n=0}^{\infty} \frac{(-5)^n}{n!} (x+10)^n$

32. $\sum_{n=10}^{\infty} n! (x+5)^n$

33. $\sum_{n=12}^{\infty} e^n (x-2)^n$

34. $\sum_{n=2}^{\infty} \frac{(x+4)^n}{(n \ln n)^2}$

In Exercises 35–40, use Eq. (2) to expand the function in a power series with center $c = 0$ and determine the interval of convergence.

35. $f(x) = \frac{1}{1-3x}$

36. $f(x) = \frac{1}{1+3x}$

37. $f(x) = \frac{1}{3-x}$

38. $f(x) = \frac{1}{4+3x}$

39. $f(x) = \frac{1}{1+x^2}$

40. $f(x) = \frac{1}{16+2x^3}$

41. Use the equalities

$$\frac{1}{1-x} = \frac{1}{-3-(x-4)} = \frac{-\frac{1}{3}}{1+\left(\frac{x-4}{3}\right)}$$

to show that for $|x-4| < 3$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{3^{n+1}}$$

42. Use the method of Exercise 41 to expand $1/(1-x)$ in power series with centers $c = 2$ and $c = -2$. Determine the interval of convergence.

43. Use the method of Exercise 41 to expand $1/(4-x)$ in a power series with center $c = 5$. Determine the interval of convergence.

44. Find a power series that converges only for x in $[2, 6)$.

45. Apply integration to the expansion

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

to prove that for $-1 < x < 1$,

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

46. Use the result of Exercise 45 to prove that

$$\ln\frac{3}{2} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$$

Use your knowledge of alternating series to find an N such that the partial sum S_N approximates $\ln\frac{3}{2}$ to within an error of at most 10^{-3}. Confirm using a calculator to compute both S_N and $\ln\frac{3}{2}$.

47. Let $F(x) = (x+1)\ln(1+x) - x$.

(a) Apply integration to the result of Exercise 45 to prove that for $-1 < x < 1$,

$$F(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{n(n+1)}$$

(b) Evaluate at $x = \frac{1}{2}$ to prove

$$\frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} = \frac{1}{1\cdot 2\cdot 2^2} - \frac{1}{2\cdot 3\cdot 2^3} + \frac{1}{3\cdot 4\cdot 2^4} - \frac{1}{4\cdot 5\cdot 2^5} + \cdots$$

(c) Use a calculator to verify that the partial sum S_4 approximates the left-hand side with an error no greater than the term a_5 of the series.

48. Prove that for $|x| < 1$,

$$\int \frac{dx}{x^4+1} = x - \frac{x^5}{5} + \frac{x^9}{9} - \cdots$$

Use the first two terms to approximate $\int_0^{1/2} dx/(x^4+1)$ numerically. Use the fact that you have an alternating series to show that the error in this approximation is at most 0.00022.

49. Use the result of Example 7 to show that

$$F(x) = \frac{x^2}{1\cdot 2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \cdots$$

is an antiderivative of $f(x) = \tan^{-1}x$ satisfying $F(0) = 0$. What is the radius of convergence of this power series?

50. Verify that function $F(x) = x\tan^{-1}x - \frac{1}{2}\log(x^2+1)$ is an antiderivative of $f(x) = \tan^{-1}x$ satisfying $F(0) = 0$. Then use the result of Exercise 49 with $x = \frac{1}{\sqrt{3}}$ to show that

$$\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\frac{4}{3} = \frac{1}{1\cdot 2(3)} - \frac{1}{3\cdot 4(3^2)} + \frac{1}{5\cdot 6(3^3)} - \frac{1}{7\cdot 8(3^4)} + \cdots$$

Use a calculator to compare the value of the left-hand side with the partial sum S_4 of the series on the right.

51. Evaluate $\sum_{n=1}^{\infty}\frac{n}{2^n}$. *Hint:* Use differentiation to show that

$$(1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1} \quad (\text{for } |x| < 1)$$

52. Use the power series for $(1+x^2)^{-1}$ and differentiation to prove that for $|x| < 1$,

$$\frac{2x}{(x^2+1)^2} = \sum_{n=1}^{\infty}(-1)^{n-1}(2n)x^{2n-1}$$

53. Show that the following series converges absolutely for $|x| < 1$ and compute its sum:

$$F(x) = 1 - x - x^2 + x^3 - x^4 - x^5 + x^6 - x^7 - x^8 + \cdots$$

Hint: Write $F(x)$ as a sum of three geometric series with common ratio x^3.

54. Show that for $|x| < 1$,

$$\frac{1+2x}{1+x+x^2} = 1 + x - 2x^2 + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \cdots$$

Hint: Use the hint from Exercise 53.

55. Find all values of x such that $\sum_{n=1}^{\infty}\frac{x^{n^2}}{n!}$ converges.

56. Find all values of x such that the following series converges:

$$F(x) = 1 + 3x + x^2 + 27x^3 + x^4 + 243x^5 + \cdots$$

57. Find a power series $P(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfying the differential equation $y' = -y$ with initial condition $y(0) = 1$. Then use Theorem 1 of Section 5.8 to conclude that $P(x) = e^{-x}$.

58. Let $C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

(a) Show that $C(x)$ has an infinite radius of convergence.

(b) Prove that $C(x)$ and $f(x) = \cos x$ are both solutions of $y'' = -y$ with initial conditions $y(0) = 1$, $y'(0) = 0$. This initial value problem has a unique solution, so we have $C(x) = \cos x$ for all x.

59. Use the power series for $y = e^x$ to show that

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

Use your knowledge of alternating series to find an N such that the partial sum S_N approximates e^{-1} to within an error of at most 10^{-3}. Confirm this using a calculator to compute both S_N and e^{-1}.

60. Let $P(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series solution to $y' = 2xy$ with initial condition $y(0) = 1$.

(a) Show that the odd coefficients a_{2k+1} are all zero.

(b) Prove that $a_{2k} = a_{2k-2}/k$ and use this result to determine the coefficients a_{2k}.

61. Find a power series $P(x)$ satisfying the differential equation

$$y'' - xy' + y = 0 \qquad \boxed{9}$$

with initial condition $y(0) = 1$, $y'(0) = 0$. What is the radius of convergence of the power series?

62. Find a power series satisfying Eq. (9) with initial condition $y(0) = 0$, $y'(0) = 1$.